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An Introduction to Supersymmetric Quantum Mechanics

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AN INTRODUCTION TO SUPERSYMMETRIC QUANTUM MECHANICS

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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In this thesis, the general framework of supersymmetric quantum mechanics and the path integral approach will be presented (as well as the worked out example of the supersymmetric harmonic oscillator). Then the theory will be specialized to the case of supersymmetric quantum mechanics on Riemannian manifolds, which will start from a supersymmetric Lagrangian for the general case and the special case for S^2. Afterwards, there will be a discussion on the superfield formalism. Concluding this thesis will be the Hamiltonian formalism followed by the inclusion of deformations by potentials.

CHAPTER 1

INTRODUCTION

Quantum mechanics is one of the most important physical theories to come out of the 20th century. Without its development, many technologies and other physical theories would simply not exist. After quantum mechanics came quantum electrodynamics (QED), the first theory to successfully marry quantum mechanics and special relativity. Then with the discovery of new particles that decay rapidly came the development of the theory responsible for the Weak force and eventually the unification with electromagnetism under the Electroweak theory. Afterward, the development of the Standard Model and quantum chromodynamics (QCD) have rounded out our understanding of particle physics. With the discovery of more fundamental particles and experiments verifying its many predictions, the Standard Model has become a pinnacle of human intellect.

Through out each of these descriptions of physical phenomena, symmetries are at the core. The unification of electromagnetism and the weak interaction is accomplished using $U(1) \times SU(2)$, while QCD is described by $SU(3)$. However, in order to develop unified physical theories, i.e. the inclusion of Einstein's theory of general relativity into the Standard Model, new symmetries must be proposed. Currently the Standard Model One of the proposed methods of unification is the use of, what is called, supersymmetry. Supersymmetry, in a nutshell, is a proposed fundamental symmetry that relates bosons and fermions. Any physical theory, classical or quantum, can incorporate supersymmerty. The framework that will be discussed in chapter 2 is at the core of any supersymmetric theory, whether it is quantum machanics or quan-

tum field theory. Referring to chapter 3 of this thesis, the main example starts with a supersymmetric Lagrangian in the classical setting and then upon quantization maintains the supersymmetry. One of the advantages of supersymmetry, from the point of view of running calculations, is that supersymmetry provides a mechanism for the cancellation of the ultraviolet divergences that arise in any realistic quantum field theory in the traditional sense [1].

Supersymmetry has a fascinating history. Before the first notion of supersymmetry, a precursor to the modern theory harkens back to the time of Schrödinger. In two of his papers, Schrödinger presents a method of solving differential equations by factorization and even solves the harmonic oscillator and non-relativistic hydrogen atom using this method [2, 3]. Schrödinger's approach at first glance looks like a mathematical trick but is actually closely related, if not equivalent, to part of the requirements of supersymmetric quantum mechanics. For example, consider the eigenvalue problem

$$(\partial_x \partial_x - 1)\,\psi = \lambda\psi. \tag{1.1}$$

Under Schödinger's factorization method, this can be factorized two ways as

$$(\partial_x + 1)(\partial_x - 1)\psi = \lambda\psi \tag{1.2}$$

$$(\partial_x - 1)(\partial_x + 1)\psi = \lambda\psi. \tag{1.3}$$

Now taking the sum of both equations, we have

$$[(\partial_x + 1)(\partial_x - 1) + (\partial_x - 1)(\partial_x + 1)]\,\psi = 2\lambda\psi. \tag{1.4}$$

We maintain this form because, in general, the two "factors" in each term may not commute; however in this example they commute. Realizing that the left hand side

is simply an anti-commutator, we have

$$\{\partial_x + 1, \partial_x - 1\} = 2\left(\partial_x \partial_x - 1\right), \tag{1.5}$$

matching part of the definition of supersymmetric quantum mechanics; that will be later defined at the beginning of Chapter 2.

Afterwards in the early 1960's, Gell-Mann and Ne'eman successfully described the relations between various strongly interacting, same spin particles of different charge and strangeness using the group $SU(3)$ [4, 5]. Later in 1967, the Coleman-Mandula theorem was proved using less restrictive assumptions [4].

The Coleman-Mandula Theorem states:

Theorem 1 *Let G be a connected symmetry group of the scattering matrix, i.e. a group whose generators commute with the scattering matrix S, and make the following five assumptions:*

1. *Lorentz invariance: G contains a subgroup which is locally isomorphc to the Poincaré group.*
2. *Particle finiteness: All particle types correspond to positive-energyrepresentations of the Poincaré group. For any finite mass M, there is only a finite number of particles with mass less than M.*
3. *Weak elastic analyticity: Elastic scattering amplitudes are analytic functions of the center-of-mass energy squared s and the invariant momentum transfer squared t in some neighborhood of the physical region, except at normal thresholds.*
4. *Occurrence of scattering: Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum*

eigenstates, and let $|p, p'\rangle$ *be the two-particle state constructed from these. Then*

$$T\,|p, p'\rangle \neq 0$$

where T *is the T-matrix defined by*

$$S = \mathbb{1} - i(2\pi)^4\delta^4(p_\mu - p'_\mu)T$$

except, perhaps, for certain isolated vaules of S*. In simpler terms this assumption means: Two plane waves scatter at almost any energy.*

5. *Techical assumption: The generators of* G*, considered as the integral operators in momentum space, have distributions for their kernels.*

Then the group G *is locally isomorphic to the direct product of a compact symmetry group group and the Poincaré group.*

Since this theorem only applies for transformations that take fermions to bosons and *vice versa*, supersymmetry is the only possibility [1, 4]. However, this fact was not immediately realized; instead supersymmetry developed independently from the Coleman-Mandula theorem in a series of papers on string theory [1, 4].

Currently, supersymmetry is used in string theory and quantum field theory in order to unify the standard model with Einstein's theory of relativity. In the experimental setting collider experiments, like the Large Hadron Collier (LHC), are on the look for the lightest supersymmetric particle (LSP). LSP is generic name given to the lightest particle in a SUSY theory. Due to constraints from cosmology, the LSP must be neutral, weakly interacting, and stable [6, 7]. The current candidates amoung the superparticles are the sneutrino, the neutalino, and the gravitino [6, 7]. The sneutrino and the neutalino are expected to have an upper mass limit of 1TeV while the gravitino's upper mass limit is much less than 1keV [6, 7, 4]

However, as of May 2019, there is no evidence confirming the existence of superpartners; which either means that supersymmetry does not exist in nature or supersymmetry is broken at the energy levels available [1]. Despite the lack of confirmation, there is one source suggesting that their data fits the behavior of stau decay into a tau lepton, however it is too early to confirm [8].

In order to understand supersymmetry, this thesis will examine some consequences of supersymmetry. This will not be a thorough examination of the subject, merely an introduction to the area in order to understand how supersymmetry works.

In Chapter 2, we begin with defining supersymmetric quantum mechanics and explore some immediate consequence. Then, we will work out the supersymmetric version of the harmonic oscillator. Finishing out the chapter will be quick a discussion of the path integral approach to quantum mechanics.

Afterward, Chapter 3 begins with working out the general case of supersymmetric quantum mechanics on Riemannian manifolds, so that general relativity can later be incorporated, and a special case for the surface of a sphere starting from a Lagrangian. Next, we will look into the superfield formalism. Then we will go through the general case and the special case of S^2 again but in Hamiltonian formalism. Concluding this thesis will be a discussion of deformations by outside potentials to the theory laid out in Chapter 3.

CHAPTER 2

SUPERSYMMETRIC QUANTUM MECHANICS

Through out this chapter, we will discuss concepts and the framework necessary for supersymmetric (SUSY) quantum mechanics. The framework presented in this chapter can be applied to any physical theory, either in the classical setting or quantum setting, and it is at core all SUSY quantum field theories (QFT).

Starting off the chapter, we will define supersymmetric quantum mechanics. Then we will work out how the bosons and fermions are related under SUSY. Next, there will a discussion of the Witten index. Afterwards, the SUSY harmonic oscillator will be worked out. Finshing out the chapter will be the path integral approach. This chapter will cover concepts from references [9, 10, 11, 12].

2.1 General Formalism

A $\mathbb{Z}_2$-graded Hilbert space of states $\mathcal{H}$ is the direct sum of two Hilbert spaces: the bosonic (even) states $\mathcal{H}^B$, and the fermionic (odd) states $\mathcal{H}^F$. With respect to the decomposition $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$, the $\mathbb{Z}_2$-grading is defined by the eigenvalues of the operator $(-1)^F$, which can be represented in block matrix form as

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.1}$$

More precisely, the $\mathbb{Z}_2$-grading behaves as

$$(-1)^F \mid_{\mathcal{H}^B} = \mathrm{Id}$$

$$(-1)^F \mid_{\mathcal{H}^F} = -\mathrm{Id}.$$

Here even and odd can also describe operators. An even (bosonic) operator means that some operator A commutes with $(-1)^F$. Conversely, odd (fermionic) operator means that some operator B anti-commutes with $(-1)^F$.

Supersymmetric quantum mechanics is a quantum theory with a positive $\mathbb{Z}_2$-graded Hilbert space of states $\mathcal{H}$ with an even operator H as the Hamiltonian and odd operator Q and $Q^\dagger$ as supercharges. These operators obey the relations:

$$Q^2 = Q^{\dagger 2} = 0 \tag{2.2}$$

$$\{Q, Q^\dagger\} = 2H, \tag{2.3}$$

where is the anti-commutator

$$\{A, B\} = AB + BA.$$

Here this is the natural commutator for a pair of odd operators. Should the operators both been even or one evan and one odd, the regular commutator would have been the natural choice. As a result, the supercharges are conserved:

$$[H, Q] = \left[H, Q^\dagger\right] = 0. \tag{2.4}$$

Hereafter $Q^\dagger$ and $\overline{Q}$ may be used interchangeably.

The Hamiltonian preserves the decomposition $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$ while the supercharges map one subspace to the other:

$$Q, Q^\dagger : \mathcal{H}^B \to \mathcal{H}^F,$$
$$Q, Q^\dagger : \mathcal{H}^F \to \mathcal{H}^B.$$

So an arbitrary vector $|v\rangle$ decomposes as

$$|v\rangle = |v^B\rangle + |v^F\rangle$$

where $|v^B\rangle \in \mathcal{H}^B$ and $|v^F\rangle \in \mathcal{H}^F$, and thus

$$H(-1)^F |v\rangle = H |v^B\rangle - H |v^F\rangle .$$

Since the Hamiltonian is even, we can factor out the $\mathbb{Z}_2$-grading to get

$$H(-1)^F |v\rangle = (-1)^F H |v\rangle . \tag{2.5}$$

Similarly without loss of generality, the supercharges act on the vector $|v\rangle$ as

$$Q(-1)^F |v\rangle = Q |v^B\rangle - Q |v^F\rangle .$$

Since the supercharges are odd, factoring out the $\mathbb{Z}_2$-grading brings in a negative sign to get

$$Q(-1)^F |v\rangle = -(-1)^F Q |v\rangle \tag{2.6}$$

$$Q^\dagger(-1)^F |v\rangle = -(-1)^F Q^\dagger |v\rangle . \tag{2.7}$$

The first consequence of how supersymmetric quantum mechanics is defined and the positive-definiteness of the Hilbert space is that the Hamiltonian, following directly from eq.(2.2), is a non-negative operator

$$H = \frac{1}{2}\{Q, Q^\dagger\} \geq 0. \tag{2.8}$$

To show this we take

$$\langle v| H |v\rangle = \frac{1}{2} \langle v| QQ^\dagger |v\rangle + \frac{1}{2} \langle v| Q^\dagger Q |v\rangle ,$$

and note that $\langle v| Q = \overline{Q^\dagger |v\rangle}$ and $\langle v| Q^\dagger = \overline{Q |v\rangle}$. Therefore

$$\langle v| H |v\rangle = \frac{1}{2}\overline{Q^\dagger |v\rangle} \cdot Q^\dagger |v\rangle + \frac{1}{2}\overline{Q |v\rangle} \cdot Q |v\rangle ,$$

where the dots are the standard dot product for Hilbert space $\mathcal{H}$. From here we can note that both terms on the right hand side are non-negative

$$\langle v| H |v\rangle = \frac{1}{2}||Q^\dagger |v\rangle ||^2 + \frac{1}{2}||Q |v\rangle ||^2 \geq 0,$$

due to the dot product of conjugate pair of vectors $|A\rangle, \overline{|A\rangle}$ is a norm, which by definition, is non-negative. A state has zero energy if and only if it is annihilated by Q and $Q^\dagger$:

$$H |\alpha\rangle = 0 \Longleftrightarrow Q |\alpha\rangle = Q^\dagger |\alpha\rangle = 0. \tag{2.9}$$

To show the backward direction is quite simple. Assume that $Q |\alpha\rangle = Q^\dagger |\alpha\rangle = 0$, then the Hamiltonian acting on a vector becomes

$$H |v\rangle = \frac{1}{2}QQ^\dagger |\alpha\rangle + \frac{1}{2}Q^\dagger Q |\alpha\rangle = 0.$$

For the forward direction, assume that $H |\alpha\rangle = 0$. Since

$$\langle \alpha| H |\alpha\rangle = \frac{1}{2} \langle \alpha|QQ^\dagger|\alpha\rangle + \frac{1}{2} \langle \alpha|Q^\dagger Q|\alpha\rangle$$

is non-negative, this implies that

$$Q |\alpha\rangle = Q^\dagger |\alpha\rangle = 0.$$

Due to the non-negativity of the Hamiltonian, a zero energy state is a ground state, i.e.

$$\langle v| H |v\rangle = E \langle v|v\rangle = E \geq 0.$$

States that are annihilated by Q or $Q^\dagger$ are states invariant under the supersymmetry and are called *supersymmetric states*. What we have seen above is that a zero energy state is a supersymmetric state and vice versa. Thus, in what follows we call such a

state a *supersymmetric ground state*.

The Hilbert space can be decomposed in terms of eigenspaces of the Hamiltonian when $\mathcal{H}_{(n)}$ is defined such that for all $n \in \mathbb{N}$

$$\mathcal{H} = \bigoplus_{n=0,1,\ldots} \mathcal{H}_{(n)} \tag{2.10}$$

$$H|_{\mathcal{H}_{(n)}} = E_n \,\mathrm{Id}|_{\mathcal{H}_{(n)}}. \tag{2.11}$$

Since $Q, Q^\dagger$, and $(-1)^F$ commute with the Hamiltonian, these operators preserve the energy levels:

$$Q, Q^\dagger, (-1)^F : \mathcal{H}_{(n)} \longrightarrow \mathcal{H}_{(n)}. \tag{2.12}$$

To show that two operators that commute preserve the eigenvalue, first assume there are two commuting operators A and B which are acting on an eigenstate of A denoted as $|A\rangle$. Since the two operators commute,

$$AB\,|A\rangle = BA\,|A\rangle\,.$$

Since the A is now acting its eigenstate,

$$AB\,|A\rangle = B\lambda\,|A\rangle\,,$$

where λ is the eigenvalue of A. Since scalar values commute with operators, the λ can be pulled out front of the operator B

$$AB\,|A\rangle = \lambda B\,|A\rangle\,,$$

thus showing that the eigenstate is preserved.

In particular, each energy level $\mathcal{H}_{(n)}$ is decomposed into even and odd (bosonic and ferimonic) subspaces

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} \oplus \mathcal{H}^F_{(n)}, \tag{2.13}$$

and the supercharges map one subspace to the other

$$Q, Q^\dagger : \mathcal{H}^B_{(n)} \longrightarrow \mathcal{H}^F_{(n)} \tag{2.14}$$

$$Q, Q^\dagger : \mathcal{H}^F_{(n)} \longrightarrow \mathcal{H}^B_{(n)}. \tag{2.15}$$

Now let us consider $Q_1 = Q + Q^\dagger$. Due to eq.(2.2), only the cross terms of squaring Q_1 survive, which by eq.(2.3) is twice the Hamiltonian.

$$Q_1^2 = QQ^\dagger + Q^\dagger Q = 2H \tag{2.16}$$

This operator preserves each energy level, mapping $\mathcal{H}^B_{(n)}$ to $\mathcal{H}^F_{(n)}$ and $\mathcal{H}^F_{(n)}$ to $\mathcal{H}^B_{(n)}$. Since $Q_1^2 = 2E_n$ at the n^{th} level, as long as $E_n > 0$, Q_1 is invertible, i.e.

$$Q_1^{-1} = \frac{Q_1}{2E_n}, \tag{2.17}$$

and defines an isomorphism

$$\mathcal{H}^B_{(n)} \cong \mathcal{H}^F_{(n)}. \tag{2.18}$$

Thus the bosonic and fermionic states are paired at each excited level. At the zero energy level $\mathcal{H}_{(0)}$, however, the operator Q_1 squares to zero and does not lead to an isomorphism. In particular the bosonic and fermionic supersymmetric ground states do not have to be paired.

Now, let us consider a continuous deformation of the theory (i.e., the spectrum of the Hamiltonian deforms continuously) while preserving supersymmetry. Here the excited states (the states with positive energy) move in bosonic/fermionic pairs due to the isomorphism discussed above. Some excited level may split into several levels but the number of bosonic and fermionic states must be the same at each of the new levels. Meanwhile, some of the zero energy states may acquire positive energy and some positive energy states may become zero energy states, but those states must

again come in pairs of bosonic and fermionic states. This means that the number of bosonic ground states minus the number of fermionic ground states is invariant. This invariant can also be represented as

$$\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)} = \mathrm{Tr}(-1)^F e^{-\beta H}. \tag{2.19}$$

For the explicit calculation refer to Appendix B. This is because in computing the trace on the right-hand side, the states with positive energy come in pairs that cancel out when weighted with $(-1)^F$ and only the ground states survive. This invariant is called the *supersymmetric index* or the *Witten index* and is sometimes also denoted by the shorthand notation Tr $(-1)^F$.

Since $Q^2 = 0$ we have a $\mathbb{Z}_2$-graded complex of vector spaces

$$\mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B \xrightarrow{Q} \mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B; \tag{2.20}$$

for more information on $\mathbb{Z}$-graded spaces, refer to Appendix A. Due to $Q^2 = 0$, this implies that when Q acts on a vector in either $\mathcal{H}^F$or $\mathcal{H}^B$, Q takes the vector to a subset of the other space, i.e.

$$\mathrm{Im}Q \subseteq \mathrm{Ker}Q \tag{2.21}$$

where ImQ is the image of Q and Ker$Q$ is the kernel of Q. Now consider the cohomology of this complex, i.e.

$$H^B(Q) := \frac{\mathrm{Ker}\, Q : \mathcal{H}^B \longrightarrow \mathcal{H}^F}{\mathrm{Im}\, Q : \mathcal{H}^F \longrightarrow \mathcal{H}^B} \tag{2.22}$$

$$H^F(Q) := \frac{\mathrm{Ker}\, Q : \mathcal{H}^F \longrightarrow \mathcal{H}^B}{\mathrm{Im}\, Q : \mathcal{H}^B \longrightarrow \mathcal{H}^F}. \tag{2.23}$$

The complex shown in eq.(2.20) decomposes into energy levels. At each of the excited levels, the complex is an exact sequence, making the cohomology vanish. This is seen

by noting that if the vector $|\alpha\rangle$ at the n^{th} level is Q-closed, $Q|\alpha\rangle = 0$, then by the relation $1 = \frac{QQ^\dagger + Q^\dagger Q}{2E_n}$ that holds on $\mathcal{H}_{(n)}$we have $|\alpha\rangle = \frac{QQ^\dagger}{2E_n}|\alpha\rangle$; namely $|\alpha\rangle$ is Q-exact. At the zero energy level $\mathcal{H}(0)$, the coboundary operator is trivial, $Q = 0$, and the cohomology is nothing but $\mathcal{H}^B$ and $\mathcal{H}^F$ themselves. Thus, we have seen that the cohomology groups come purely from the supersymmetric ground states

$$\mathcal{H}^B(Q) = \mathcal{H}^B_{(0)};\ \mathcal{H}^F(Q) = \mathcal{H}^F_{(0)}. \tag{2.24}$$

In other words, the space of supersymmetric ground states is characterized as the cohomology of the Q-operator.

So far, we have assumed only the $\mathbb{Z}_2$-grading denoted by $(-1)^F$. Note that in some cases there can be a finer grading such as a $\mathbb{Z}$-grading that reduces modulo 2 to the $\mathbb{Z}_2$-grading under consideration. Such is the case if there is a Hermitian operator F with integral eigenvalues such that $e^{i\pi F} = (-1)^F$. The Hilbert space $\mathcal{H}$ can be decomposed with respect to the eigenspaces of F as $\mathcal{H} = \oplus_{p\in\mathbb{Z}}\mathcal{H}^p$ and the bosonic and fermionic subspaces are simply $\mathcal{H}^B = \oplus_{p\text{even}}\mathcal{H}^p$ and $\mathcal{H}^F = \oplus_{p\text{odd}}\mathcal{H}^p$. Furthermore, if Q has charge 1, i.e.

$$[F, Q] = Q, \tag{2.25}$$

the $\mathbb{Z}_2$-graded complex shown in Eq.(2.20) splits into a $\mathbb{Z}$-graded complex

$$\ldots \xrightarrow{Q} \mathcal{H}^{p-1} \xrightarrow{Q} \mathcal{H}^p \xrightarrow{Q} \mathcal{H}^{p+1} \xrightarrow{Q} \ldots; \tag{2.26}$$

refer to Appendix A for an explicit derivation of eq.(2.26). There is also a cohomology group for each $p \in \mathbb{Z}$:

$$\mathcal{H}^p(Q) = \frac{\text{Ker}Q : \mathcal{H}^p \to \mathcal{H}^{p+1}}{\text{Im}Q : \mathcal{H}^{p-1} \to \mathcal{H}^p} \tag{2.27}$$

Of course, the space of supersymmetric ground states is the sum of these cohomology

groups and the bosonic/fermionic decomposition corresponds to

$$\mathcal{H}^B_{(0)} = \bigoplus_{p\,\text{even}} \mathcal{H}^p(Q),\ \mathcal{H}^F_{(0)} = \bigoplus_{p\,\text{odd}} \mathcal{H}^p(Q). \tag{2.28}$$

The Witten index is then the Euler characteristic of the complex

$$\text{Tr}(-1)^F = \sum_{p\in\mathbb{Z}} (-1)^p \dim H^p(Q). \tag{2.29}$$

It is possible to generalize this $\mathbb{Z}_2$-grading to the case with a $\mathbb{Z}_{2k}$-grading. However this is beyond the scope of this thesis and will be left as an exercise for the reader.

2.2 Example: The SUSY Harmonic Oscillator

In the particular case of the SUSY harmonic oscillator, the Hilbert space decomposes as

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F, \tag{2.30}$$

with

$$\mathcal{H}^B = L^2(\mathbb{R}, \mathbb{C})\,|0\rangle \tag{2.31}$$

$$\mathcal{H}^F = L^2(\mathbb{R}, \mathbb{C})\,\overline{\psi}\,|0\rangle\,, \tag{2.32}$$

where $L^2(\mathbb{R}, \mathbb{C})$ is the Hilbert space of the bosonic harmonic oscillator, on which H_{osc} acts non-trivially and $\mathbb{C}^2 := \mathbb{C}\,|0\rangle \oplus \mathbb{C}\,\overline{\psi}\,|0\rangle$ is the space on which H_F acts non-trivially. In the $\{|0\rangle\,, \overline{\psi}\,|0\rangle$ basis, the the two states are represented as

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \overline{\psi}\,|0\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.33}$$

Suppose the Hamiltonian for our quantum mechanical system is

$$H = \frac{1}{2}\left(p^2 + \omega^2 x^2 + \omega\left[\overline{\psi}, \psi\right]\right). \tag{2.34}$$

To check that this Hamiltonian is supersymmetric, consider the supercharges

$$Q = \overline{\psi}\left(ip + \omega x\right) \tag{2.35}$$

$$Q^\dagger = \psi\left(-ip + \omega x\right), \tag{2.36}$$

where $\psi, \overline{\psi}$ are fermionic (odd) variables that augment physical space to describe fermions, and the canonical commutation relations

$$[x, p] = i \tag{2.37}$$

$$\{\psi, \overline{\psi}\} = 1, \tag{2.38}$$

where p and x operators

$$p = -i\partial_x \tag{2.39}$$

$$x = x\cdot, \tag{2.40}$$

where the dot represents the act of multiplication, act on $f \in L^2(\mathbb{R}, \mathbb{C})$. Using the anti-commutator of the supercharges, we have

$$\{Q, Q^\dagger\} = \overline{\psi}\psi\left(p^2 + \omega^2 x^2 + i\omega px - i\omega xp\right) + \psi\overline{\psi}\left(p^2 + \omega^2 x^2 - i\omega px + i\omega xp\right). \tag{2.41}$$

Using the commutation relation from eq.(2.37), we have

$$\{Q, Q^\dagger\} = \overline{\psi}\psi\left(p^2 + \omega^2 x^2 + \omega\right) + \psi\overline{\psi}\left(p^2 + \omega^2 x^2 - \omega\right). \tag{2.42}$$

Regrouping this by like terms, we can say that

$$\{Q, Q^\dagger\} = \left(p^2 + \omega^2 x^2\right)\left(\overline{\psi}\psi + \psi\overline{\psi}\right) + \omega\left(\overline{\psi}\psi - \psi\overline{\psi}\right). \tag{2.43}$$

Now using the anti-commutation relation from eq.(2.38), we get twice the Hamiltonian.

$$\{Q, Q^\dagger\} = \left(p^2 + \omega^2 x^2\right) + \omega\left[\overline{\psi}, \psi\right] \tag{2.44}$$

Now in order to determinw the eigenvalues/energy states for this system, first define

$$H_B = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) \tag{2.45}$$

$$H_F = \omega\left[\overline{\psi}, \psi\right]. \tag{2.46}$$

Since this two pieces of the Hamiltonian commute with each other, they share common eigenstates. Therefore the eigenvalues for each part can be found using the same eigenstates. So for H_B it is commonly known that for the harmonic oscillator that the energy states are

$$E_n^B = E_n^{osc} = |\omega|\left(n + \frac{1}{2}\right), \tag{2.47}$$

where $n \in \mathbb{Z}_{\geq 0}$; for the explicit calculation of the ordinary harmonic oscillator energy values refer to Appendix C. Now for the fermionic part, using the matrix representation of ψ and $\overline{\psi}$ as

$$\psi \doteq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overline{\psi} \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.48}$$

in the $\{|0\rangle, \overline{\psi}|0\rangle\}$ basis, the Hamiltonian for the fermionic part becomes

$$H_F = \frac{\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.49}$$

From here, it is clear that the two eigenvalues are

$$E_F = \pm\frac{|\omega|}{2} \tag{2.50}$$

since $\omega = \pm\sqrt{k/m}$.

Now looking at the combined spectra yields

$$E_F + E_n^B = |\omega| n \tag{2.51}$$

where $n \in \mathbb{Z}_{\geq 0}$. From here it is easy to see that there is a supersymmetric ground state at $n = 0$.

We now calculate the partition function and the Witten index. Given the factorization of the Hilbert, the partition function and the Witten index are given by

$$Z(\beta) := \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\beta H_B} \cdot \mathrm{Tr}\, e^{-\beta H_F} \tag{2.52}$$

$$\mathrm{Tr}\,(-1)^F := \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{Tr}\, e^{-\beta H_B} \cdot \mathrm{Tr}\left[(-1)^F e^{-\beta H_F}\right]. \tag{2.53}$$

Now calculating the individual parts, we can use the eigenvalues from eq.(2.47) and eq.(2.50) to get

$$\mathrm{Tr}\, e^{-\beta H_B} = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)|\omega|} \tag{2.54}$$

$$\mathrm{Tr}\, e^{-\beta H_F} = e^{-\beta\omega/2} + e^{\beta\omega/2} \tag{2.55}$$

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H_F}\right] = e^{-\beta\omega/2} - e^{\beta\omega/2}. \tag{2.56}$$

To evaluate the infinite sum in eq.(2.54), we can rewrite the summation as

$$\mathrm{Tr}\, e^{-\beta H_B} = e^{-\frac{1}{2}\beta|\omega|} \sum_{n=0}^{\infty} e^{-\beta|\omega|n/2}. \tag{2.57}$$

Since this is a infinite geometric series where $e^{-\beta|\omega|} < 1$, the summation becomes

$$\mathrm{Tr}\, e^{-\beta H_B} = \frac{e^{-\frac{1}{2}\beta|\omega|}}{1 - e^{-\beta|\omega|}}. \tag{2.58}$$

This reduces to

$$\mathrm{Tr}\, e^{-\beta H_B} = \frac{1}{e^{\beta|\omega|/2} - e^{-\beta|\omega|/2}}. \tag{2.59}$$

Using eq.(2.55), eq.(2.56), and eq.(2.59), the partition function and the Witten index become

$$Z(\beta) = \frac{e^{-\beta\omega/2} + e^{\beta\omega/2}}{e^{\beta|\omega|/2} - e^{-\beta|\omega|/2}} \tag{2.60}$$

$$\mathrm{Tr}\,(-1)^F = \frac{e^{-\beta\omega/2} - e^{\beta\omega/2}}{e^{\beta|\omega|/2} - e^{-\beta|\omega|/2}}. \tag{2.61}$$

By definition the partition function reduces to hyperbolic cotangent

$$Z(\beta) = \coth\left(\frac{\beta|\omega|}{2}\right), \tag{2.62}$$

Then by examining the cases when $\omega > 0$ and $\omega < 0$, the numerator and denominator only differ by a sign, i.e.

$$\mathrm{Tr}\,(-1)^F = \pm 1. \tag{2.63}$$

Note that the partition function depends on the circumference β of S^1 whereas the supersymmetric index does not.

Now calculating the cohomology for the SUSY harmonic oscillator, we let Q act

on an arbitrary vector $|v\rangle = f|0\rangle + g\overline{\psi}|0\rangle$ to get

$$Q|v\rangle = \overline{\psi}(ip + \omega x) f|0\rangle + \overline{\psi}(ip + \omega x) g\overline{\psi}|0\rangle . \tag{2.64}$$

Since the fermionic variables annihilate themselves, we have

$$Q|v\rangle = \overline{\psi}(ip + \omega x) f|0\rangle . \tag{2.65}$$

This means that

$$\mathrm{Im}\, Q : \mathcal{H}^F \longrightarrow \mathcal{H}^B = \phi \tag{2.66}$$

$$\mathrm{Im}\, Q : \mathcal{H}^B \longrightarrow \mathcal{H}^F = L^2(\mathbb{R}, \mathbb{C}) \tag{2.67}$$

$$\mathrm{Ker}\, Q : \mathcal{H}^F \longrightarrow \mathcal{H}^B = L^2(\mathbb{R}, \mathbb{C}) \tag{2.68}$$

$$\mathrm{Ker}\, Q : \mathcal{H}^B \longrightarrow \mathcal{H}^F = \left\{ f \in L^2(\mathbb{R}, \mathbb{C}) : (ip + \omega x) f = 0 \right\}, \tag{2.69}$$

where ϕ is the empty set and $p = -i\partial_x$ under the canonical quantization. From here we can see that

$$f(x) = A\, e^{-\frac{1}{2}\omega x^2}, \tag{2.70}$$

where A is a constant, and the cohomology is one dimensional and concentrated in even parity.

$$H^B(Q) = \mathbb{C} \tag{2.71}$$

$$H^F(Q) = 0 \tag{2.72}$$

2.3 The Path Integral Approach

The independence of the supersymmetric index from β can be exploited to relate it to computations done in zero-dimensional QFT. Namely we consider the limit $\beta \to 0$, in which case in the path-integral computation, only the time independent

modes contribute, and we are left with a finite-dimensional integral that is exactly the same integral found in the context of the zero-dimensional QFT. This also explains why the Witten index is equal to the partition function for the supersymmetric system considered for the zero-dimensional QFT.

Ordinarily, the Feynman path integral version of the partition function is

$$Z = \int \mathcal{D}X \, e^{iS(X)} \tag{2.73}$$

where $S(X)$ is the action

$$S(X) = \int L \, dt. \tag{2.74}$$

Since $S(X)$ is real, we are summing up phases associated with different paths and the convergence of the integral is a subtle problem. By considering the "Euclidean theory", we can avoid dealing with the different phases. This is obtained by "Euclideanizing" the time coordinate t by the so-called Wick rotation:

$$t \longrightarrow -i\tau. \tag{2.75}$$

Then the action $S(X) \longrightarrow iS_E(X)$, where $S_E(X)$ is the Euclidean action. This in turn defines the Euclidean partition function to be

$$Z_E = \int \mathcal{D}X \, e^{-S_E(X)}. \tag{2.76}$$

Another way to think about the Wick rotation is that, in the setting of QFT, in which uses a Lorentzian (Minkowski) metric, this will change the geometry of the space-time from Lorentzian to Euclidean. This in turn makes the geometry and calculations easier to held.

Now suppose there is a supersymmetric quantum mechanics which comes from a supersymmetric Lagrangian. Then the Witten index $\mathrm{Tr}(-1)^F e^{-\beta H}$ and the partition

function $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ on a circle of circumference β can be define in terms of a "Euclideanized" path integral as

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{AP}} e^{-S\left(X,\psi,\overline{\psi}\right)}, \tag{2.77}$$

$$\mathrm{Tr}(-1)^F = \mathrm{Tr}(-1)^F e^{-\beta H} = \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\overline{\psi}|_{\mathrm{P}} e^{-S\left(X,\psi,\overline{\psi}\right)}, \tag{2.78}$$

where the AP and P on the measure represents the use of antiperiodic and periodic boundary conditions on the fermionic fields:

$$\mathrm{AP} : \psi(0) = -\psi(\beta),\ \overline{\psi}(0) = -\overline{\psi}(\beta), \tag{2.79}$$

$$\mathrm{P} : \psi(0) = +\psi(\beta),\ \overline{\psi}(0) = +\overline{\psi}(\beta). \tag{2.80}$$

The fact that inserting $(-1)^F$ operator corresponds to changing the boundary conditions on fermions follows from the fact that fermions anti-commute with $(-1)^F$. So before the trace is taken, the fermions are multiplied by an extra minus sign. What is not completely obvious is that without the insertion of $(-1)^F$ the fermions have anti-periodic boundary condition along the circle. To understand this, let us consider the correlation functions on the circle with insertions of fermions. Due to the fermion number symmetry, the number of ψ insertions must be the same as the number of $\overline{\psi}$ insertions for the correlators to be non-vanishing. We consider the simplest case with the insertion of $\overline{\psi}(t_1)$ and $\psi(t_2)$. Let us start with $t_2 = 0 < t_1 < \beta$, and increase t_2 so that it passes through t_1 and "comes back" to β. Due to the anti-commutativity of the fermionic operators, when t_2 passes through t_1, the correlation function receives an extra minus sign. Thus, the ordinary correlation function $\langle\overline{\psi}(t_1)\psi(t_2)\rangle_{S^1_\beta}$, which corresponds to the trace without $(-1)^F$, is antiperiodic under the shift $t_2 \to t_2 + \beta$.

We saw in the operator representation that $\mathrm{Tr}\,(-1)^F e^{\beta H}$ is independent of β. What this means in this context is that in the path-integral representation on a circle

of radius β with periodic boundary conditions, the path-integral is independent of the radius of the circle. One can directly see this in the path-integral language as well. Namely, the change of the circumference is equivalent to insertion of H in the path-integral. This can in turn be viewed as the Q variation of the field $Q^\dagger$ (in view of the commutation relation $\{Q, Q^\dagger\} = 2H$). For periodic boundary conditions on the circle, Q is a symmetry of the path-integral (this only exists for periodic boundary conditions for fermions because there is no constant non-trivial ϵ that is anti-periodic along S^1). And as in our discussion in the context of zero-dimensional QFT, the correlators that are variations of fields under symmetry operations are zero. Thus the insertion of H in the path-integral gives zero, which is equivalent to β independence of the Witten index in the path-integral representation.

CHAPTER 3

SUSY QM ON RIEMANNIAN MANIFOLDS

We now specialize the general framework discussed in chapter 2 to the important example of SUSY quantum mechanics constructed out of geometric data; more specifically, the supersymmetric sigma model. This section will cover quantum mechanics concepts from references [9, 10, 11, 12, 1, 4] and concepts from differential geometry from references [9, 13, 14].

In the following sections, an in-depth look of a supersymmetric sigma model, as well as some worked on examples are presented. In particular, we will start off showing that a Lagrangian for a Riemannian manifold is supersymmetric under some SUSY transformations and derive the supercharges and Noether charge. Then we will specialize these calculation to the special case of S^2. Afterward, there will be a discussion on the superfield formalism in order to understand how to generate supersymmetric Lagrangians and SUSY transformations. Next, we will quantize both the general and S^2 cases and examine the Hamiltonians. Finally, we will include deformations by outside potentials into the theory laid out in this chapter.

3.1 The General Case

So consider the Lagrangian

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{i}{2} g_{IJ} \left(\overline{\psi}^I D_t \psi^J - D_t \overline{\psi}^I \psi^J \right) + \frac{1}{2} R_{IJKL} \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L \tag{3.1}$$

where the covariant time derivative is

$$D_t \psi^I = \partial_t \psi^I + \Gamma^I_{\alpha\beta} \dot{\phi}^\alpha \psi^\beta, \tag{3.2}$$

with $\Gamma^I_{\alpha\beta}$ being the Christoffel symbol defined as

$$\Gamma^I_{\alpha\beta} = \frac{1}{2} g^{I\delta} \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right). \tag{3.3}$$

Under the supersymmetry transformations

$$\delta\phi^I = \epsilon \overline{\psi}^I - \bar{\epsilon}\psi^I \tag{3.4}$$

$$\delta\psi^I = \epsilon \left(i\dot{\phi}^I - \Gamma^I_{\alpha\beta} \overline{\psi}^\alpha \psi^\beta \right) \tag{3.5}$$

$$\delta\overline{\psi}^I = \bar{\epsilon} \left(-i\dot{\phi}^I - \Gamma^I_{\alpha\beta} \overline{\psi}^\alpha \psi^\beta \right), \tag{3.6}$$

where ϵ and $\bar{\epsilon}$ are fermionic numbers, the action is invariant

$$\delta S = \delta \int L \, dt = 0, \tag{3.7}$$

making the classical system supersymmetric. To show that this is true, we assume a symmetric metric tensor$g_{IJ} = g_{JI}$ and normal coordinates i.e. the expression involving the metric are calculated in coordinates such that $\partial_K g_{IJ} = 0$ (so that in particular all Christoffel symbols vanish too). More precisely this is assumed at a given (but arbitrary) point, in particular higher derivatives of the metric cannot be assumed to vanish.

For brevity we can take the variation of the Lagrangian so that there are only ϵ terms since the $\bar{\epsilon}$ terms will mirror the same process through out the following calculations. Therefore the variation of the Lagrangian with respect to ϵ is

$$\begin{aligned} \delta L = {} & \frac{1}{2} g_{IJ} \delta\dot{\phi}^I \dot{\phi}^J + \frac{1}{2} g_{IJ} \dot{\phi}^I \delta\dot{\phi}^J + \\ & + \frac{i}{2} g_{IJ} \left[\overline{\psi}^I \partial_t \delta\psi^J + \overline{\psi}^I \partial_\gamma \Gamma^J_{\alpha\beta} \delta\phi^\gamma \dot{\phi}^\alpha \psi^\beta - \partial_\gamma \Gamma^I_{\alpha\beta} \delta\phi^\gamma \dot{\phi}^\alpha \overline{\psi}^\beta \psi^J - \partial_t \overline{\psi}^I \delta\psi^J \right] \\ & + \frac{1}{2} \partial_\gamma R_{IJKL} \delta\phi^\gamma \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L + \frac{1}{2} R_{IJKL} \left(\delta\psi^I \overline{\psi}^J \psi^K \overline{\psi}^L + \psi^I \overline{\psi}^J \delta\psi^K \overline{\psi}^L \right). \end{aligned} \tag{3.8}$$

Here we can rearrange the variation factors so that they are the leading factor in the

four fermion terms and multiply both side by two.

$$\begin{aligned} 2\delta L = {} & g_{IJ}\delta\dot{\phi}^I\dot{\phi}^J + g_{IJ}\dot{\phi}^I\delta\dot{\phi}^J + \\ & + ig_{IJ}\left[\overline{\psi}^I\partial_t\delta\psi^J + \overline{\psi}^I\partial_\gamma\Gamma^J_{\alpha\beta}\delta\phi^\gamma\dot{\phi}^\alpha\psi^\beta - \partial_\gamma\Gamma^I_{\alpha\beta}\delta\phi^\gamma\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J - \partial_t\overline{\psi}^I\delta\psi^J\right] \\ & + \partial_\gamma R_{IJKL}\delta\phi^\gamma\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L + R_{IJKL}\left(\delta\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L + \delta\psi^K\psi^I\overline{\psi}^J\overline{\psi}^L\right). \end{aligned} \tag{3.9}$$

Now using the ϵ part of the supersymmetry transformations

$$\begin{aligned} 2\delta L = {} & \epsilon g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + \epsilon g_{IJ}\dot{\phi}^I\partial_t\overline{\psi}^J + ig_{IJ}\left[\overline{\psi}^I\epsilon\partial_t\left(i\dot{\phi}^J - \Gamma^J_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\right)\right. \\ & \left. + \overline{\psi}^I\partial_\gamma\Gamma^J_{\alpha\beta}\epsilon\overline{\psi}^\gamma\dot{\phi}^\alpha\psi^\beta - \epsilon\partial_\gamma\Gamma^I_{\alpha\beta}\overline{\psi}^\gamma\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J - i\partial_t\overline{\psi}^I\epsilon\dot{\phi}^J\right] \\ & + \partial_\gamma R_{IJKL}\epsilon\overline{\psi}^\gamma\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L + \epsilon R_{IJKL}\left[i\dot{\phi}^I\overline{\psi}^J\psi^K\overline{\psi}^L + i\dot{\phi}^K\psi^I\overline{\psi}^J\overline{\psi}^L\right], \end{aligned} \tag{3.10}$$

we can separate the terms into three cases based on the number of fermionic variables: 1, 3, and 5. Doing this we get

$$1\psi : 2\delta L = \epsilon g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + \epsilon g_{IJ}\dot{\phi}^I\partial_t\overline{\psi}^J - g_{IJ}\overline{\psi}^I\epsilon\ddot{\phi}^J + g_{IJ}\partial_t\overline{\psi}^I\epsilon\dot{\phi}^J \tag{3.11}$$

$$\begin{aligned} 3\psi : 2\delta L = {} & ig_{IJ}\left[-\overline{\psi}^I\epsilon\partial_\gamma\Gamma^J_{\alpha\beta}\dot{\phi}^\gamma\overline{\psi}^\alpha\psi^\beta + \overline{\psi}^I\partial_\gamma\Gamma^J_{\alpha\beta}\epsilon\overline{\psi}^\gamma\dot{\phi}^\alpha\psi^\beta - \epsilon\partial_\gamma\Gamma^I_{\alpha\beta}\overline{\psi}^\gamma\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J\right] \\ & + i\epsilon R_{IJKL}\left[\dot{\phi}^I\overline{\psi}^J\psi^K\overline{\psi}^L + \dot{\phi}^K\psi^I\overline{\psi}^J\overline{\psi}^L\right] \end{aligned} \tag{3.12}$$

$$5\psi : 2\delta L = \partial_\gamma R_{IJKL}\epsilon\overline{\psi}^\gamma\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L. \tag{3.13}$$

When we bring ϵ out front for each term, it picks up a negative sign for every other fermionic variable it crosses.

$$1\psi : 2\delta L = \epsilon g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + \epsilon g_{IJ}\dot{\phi}^I\partial_t\overline{\psi}^J + \epsilon g_{IJ}\overline{\psi}^I\ddot{\phi}^J - \epsilon g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J \tag{3.14}$$

$$3\psi : 2\delta L = ig_{IJ}\left[\epsilon\overline{\psi}^I\partial_\gamma\Gamma^J_{\alpha\beta}\dot{\phi}^\gamma\overline{\psi}^\alpha\psi^\beta - \epsilon\overline{\psi}^I\partial_\gamma\Gamma^J_{\alpha\beta}\overline{\psi}^\gamma\dot{\phi}^\alpha\psi^\beta - \epsilon\partial_\gamma\Gamma^I_{\alpha\beta}\overline{\psi}^\gamma\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J\right] + i\epsilon R_{IJKL}\left[\dot{\phi}^I\overline{\psi}^J\psi^K\overline{\psi}^L + \dot{\phi}^K\psi^I\overline{\psi}^J\overline{\psi}^L\right] \tag{3.15}$$

$$5\psi : 2\delta L = \partial_\gamma R_{IJKL}\epsilon\overline{\psi}^\gamma\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L \tag{3.16}$$

. Now dealing with the one fermion part in equation (3.15), we have

$$1\psi : 2\delta L = \epsilon\left(g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + g_{IJ}\dot{\phi}^I\partial_t\overline{\psi}^J + g_{IJ}\overline{\psi}^I\ddot{\phi}^J - g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J\right). \tag{3.17}$$

By noting that the metric tensor is symmetric and that fermionic variables can commute with bosonic variables, the second and forth terms cancel.

$$1\psi : 2\delta L = \epsilon\left(g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I\ddot{\phi}^J\right). \tag{3.18}$$

Then by noting the total time derivative of $g_{IJ}\overline{\psi}^I\dot{\phi}^J$ is

$$\partial_t\left(g_{IJ}\overline{\psi}^I\dot{\phi}^J\right) = \partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\dot{\phi}^J + g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I\ddot{\phi}^J, \tag{3.19}$$

where $\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\dot{\phi}^J = 0$, we have that

$$1\psi : 2\delta L = \epsilon\left(g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I\ddot{\phi}^J\right) = 0. \tag{3.20}$$

Now moving on to the three fermion part in eq.(3.16), let us expand the Christoffel symbols using the definition in eq.(3.3).

$$\begin{aligned} 3\psi : 2\delta L = \frac{i}{2}\epsilon g_{IJ}\Big[&\overline{\psi}^I g^{J\delta}\partial_\gamma\left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}\right)\dot{\phi}^\gamma\overline{\psi}^\alpha\psi^\beta \\ &- \overline{\psi}^I g^{J\delta}\partial_\gamma\left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}\right)\overline{\psi}^\gamma\dot{\phi}^\alpha\psi^\beta \\ &- g^{I\delta}\partial_\gamma\left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}\right)\overline{\psi}^\gamma\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J\Big] \\ &+ i\epsilon R_{IJKL}\left(\dot{\phi}^I\overline{\psi}^J\psi^K\overline{\psi}^L + \dot{\phi}^K\psi^I\overline{\psi}^J\overline{\psi}^L\right). \end{aligned} \tag{3.21}$$

Now summing over J in the first two terms and in I in the third term, we have

$$\begin{aligned} 3\psi : 2\delta L = \frac{i}{2}\epsilon \Big[& \overline{\psi}^I g_I^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \dot{\phi}^\gamma \overline{\psi}^\alpha \psi^\beta \\ & - \overline{\psi}^I g_I^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \overline{\psi}^\gamma \dot{\phi}^\alpha \psi^\beta \\ & - g_J^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \overline{\psi}^\gamma \dot{\phi}^\alpha \overline{\psi}^\beta \psi^J \Big] \\ & + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \end{aligned} \tag{3.22}$$

Now summing over the remaining lower indices of the leading metric tensor in the first three terms gives

$$\begin{aligned} 3\psi : 2\delta L = \frac{i}{2}\epsilon \Big[& \overline{\psi}^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \dot{\phi}^\gamma \overline{\psi}^\alpha \psi^\beta \\ & - \overline{\psi}^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \overline{\psi}^\gamma \dot{\phi}^\alpha \psi^\beta \\ & - \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \overline{\psi}^\gamma \dot{\phi}^\alpha \overline{\psi}^\beta \psi^\delta \Big] \\ & + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \end{aligned} \tag{3.23}$$

Then by noting that the $\overline{\psi}$ factors anti-commute in conjunction with that facts that the order of derivatives does not matter and the metric tensor is symmetric, we have

$$\begin{aligned} 3\psi : 2\delta L = \frac{i}{2}\epsilon \Big[& \overline{\psi}^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta} \right) \dot{\phi}^\gamma \overline{\psi}^\alpha \psi^\beta - \overline{\psi}^\delta \partial_\gamma \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} \right) \overline{\psi}^\gamma \dot{\phi}^\alpha \psi^\beta \\ & - \partial_\gamma \left(\partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta} \right) \overline{\psi}^\gamma \dot{\phi}^\alpha \overline{\psi}^\beta \psi^\delta \Big] + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \end{aligned} \tag{3.24}$$

For convenience, we can rearrange the factors in the first three terms, without a change of sign, to get

$$\begin{aligned} 3\psi : 2\delta L = \frac{i}{2}\epsilon \Big[& \partial_\gamma \left(\underline{\partial_\alpha g_{\beta\delta}} - \partial_\delta g_{\alpha\beta} \right) \dot{\phi}^\gamma \overline{\psi}^\delta \overline{\psi}^\alpha \psi^\beta - \partial_\gamma \left(\underline{\partial_\alpha g_{\beta\delta}} + \partial_\beta g_{\delta\alpha} \right) \dot{\phi}^\alpha \overline{\psi}^\delta \overline{\psi}^\gamma \psi^\beta \\ & - \partial_\gamma \left(\partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta} \right) \dot{\phi}^\alpha \overline{\psi}^\gamma \overline{\psi}^\beta \psi^\delta \Big] + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \end{aligned} \tag{3.25}$$

Note in the previous equation the two underlined terms. Since the order of derivatives

does not matter, these two terms cancel. This leaves us with

$$3\psi : 2\delta L = \frac{i}{2}\epsilon \Big[-\underline{\partial_\gamma \partial_\delta g_{\alpha\beta} \dot{\phi}^\gamma \overline{\psi}^\delta \overline{\psi}^\alpha \psi^\beta} - \partial_\gamma \partial_\beta g_{\delta\alpha} \dot{\phi}^\alpha \overline{\psi}^\delta \overline{\psi}^\gamma \psi^\beta \\ -\partial_\gamma \left(\underline{\partial_\alpha g_{\beta\delta}} - \partial_\delta g_{\alpha\beta} \right) \dot{\phi}^\alpha \overline{\psi}^\gamma \overline{\psi}^\beta \psi^\delta \Big] + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \tag{3.26}$$

Again note in the previous equation the two underlined terms. Since the order of derivatives does not matter, these two terms add up. This leaves us with

$$3\psi : 2\delta L = \frac{i}{2}\epsilon \left(-2\partial_\gamma \partial_\delta g_{\alpha\beta} \dot{\phi}^\gamma \overline{\psi}^\delta \overline{\psi}^\alpha \psi^\beta - \underline{\partial_\gamma \partial_\beta g_{\delta\alpha} \dot{\phi}^\alpha \overline{\psi}^\delta \overline{\psi}^\gamma \psi^\beta} + \underline{\partial_\gamma \partial_\delta g_{\alpha\beta} \dot{\phi}^\alpha \overline{\psi}^\gamma \overline{\psi}^\beta \psi^\delta} \right) \\ + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \tag{3.27}$$

Once more, note in the previous equation the two underlined terms. Swapping the order for the $\overline{\psi}$ in the second underlined term gives us a negative, allowing for these two terms add up. This leaves us with

$$3\psi : 2\delta L = -i\epsilon \left(\partial_\gamma \partial_\delta g_{\alpha\beta} \dot{\phi}^\gamma \overline{\psi}^\delta \overline{\psi}^\alpha \psi^\beta + \partial_\gamma \partial_\beta g_{\delta\alpha} \dot{\phi}^\alpha \overline{\psi}^\delta \overline{\psi}^\gamma \psi^\beta \right) \\ + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^J \psi^K \overline{\psi}^L + \dot{\phi}^K \psi^I \overline{\psi}^J \overline{\psi}^L \right). \tag{3.28}$$

For convenience, we can swap the order of the factors in $\overline{\psi}^J \psi^K \overline{\psi}^L$ and $\psi^I \overline{\psi}^J \overline{\psi}^L$ in order to not pick up a negative sign, i.e. make an even number of permutations to the fermionic variables.

$$3\psi : 2\delta L = -i\epsilon \left(\partial_\gamma \partial_\delta g_{\alpha\beta} \dot{\phi}^\gamma \overline{\psi}^\delta \overline{\psi}^\alpha \psi^\beta + \partial_\gamma \partial_\beta g_{\delta\alpha} \dot{\phi}^\alpha \overline{\psi}^\delta \overline{\psi}^\gamma \psi^\beta \right) \\ + i\epsilon R_{IJKL} \left(\dot{\phi}^I \overline{\psi}^L \overline{\psi}^J \psi^K + \dot{\phi}^K \overline{\psi}^J \overline{\psi}^L \psi^I \right). \tag{3.29}$$

Now for the last term, the Riemann curvature tensor is defined, in terms of derivatives

of the metric tensor, as

$$R_{IJKL} = \frac{1}{2}\left(\partial_J\partial_K g_{IL} + \partial_I\partial_L g_{JK} - \partial_J\partial_L g_{IK} - \partial_I\partial_K g_{JL}\right) + g_{\alpha\beta}\left(\Gamma^\alpha_{JK}\Gamma^\beta_{IL} - \Gamma^\alpha_{JL}\Gamma^\beta_{IK}\right). \tag{3.30}$$

Since we are using normal coordinates, the Christoffel symbol terms disappear. This makes the three fermion part become

$$3\psi : 2\delta L = -i\epsilon\left[\partial_\gamma\partial_\delta g_{\alpha\beta}\dot{\phi}^\gamma\overline{\psi}^\delta\overline{\psi}^\alpha\psi^\beta + \partial_\gamma\partial_\beta g_{\delta\alpha}\dot{\phi}^\alpha\overline{\psi}^\delta\overline{\psi}^\gamma\psi^\beta\right] + \frac{i}{2}\epsilon\left(\partial_J\partial_K g_{IL} + \partial_I\partial_L g_{JK} - \partial_J\partial_L g_{IK} - \partial_I\partial_K g_{JL}\right)\left(\dot{\phi}^I\overline{\psi}^L\overline{\psi}^J\psi^K + \dot{\phi}^K\overline{\psi}^J\overline{\psi}^L\psi^I\right). \tag{3.31}$$

Now since the $\overline{\psi}$ factors anti-commute, the last two terms of the curvature tensor become zero.

$$3\psi : 2\delta L = -i\epsilon\left[\partial_\gamma\partial_\delta g_{\alpha\beta}\dot{\phi}^\gamma\overline{\psi}^\delta\overline{\psi}^\alpha\psi^\beta + \partial_\gamma\partial_\beta g_{\delta\alpha}\dot{\phi}^\alpha\overline{\psi}^\delta\overline{\psi}^\gamma\psi^\beta\right] + \frac{i}{2}\epsilon\left(\partial_J\partial_K g_{IL} + \partial_I\partial_L g_{JK}\right)\left(\dot{\phi}^I\overline{\psi}^L\overline{\psi}^J\psi^K + \dot{\phi}^K\overline{\psi}^J\overline{\psi}^L\psi^I\right). \tag{3.32}$$

By expanding this out

$$3\psi : 2\delta L = -i\,\epsilon\left[\partial_\gamma\partial_\delta g_{\alpha\beta}\dot{\phi}^\gamma\overline{\psi}^\delta\overline{\psi}^\alpha\psi^\beta + \partial_\gamma\partial_\beta g_{\delta\alpha}\dot{\phi}^\alpha\overline{\psi}^\delta\overline{\psi}^\gamma\psi^\beta\right] + \frac{i}{2}\epsilon\left[\left(\underline{\partial_J\partial_K g_{IL}} + \partial_I\partial_L g_{JK}\right)\dot{\phi}^I\overline{\psi}^L\overline{\psi}^J\psi^K + \left(\partial_J\partial_K g_{IL} + \underline{\partial_I\partial_L g_{JK}}\right)\dot{\phi}^K\overline{\psi}^J\overline{\psi}^L\psi^I\right], \tag{3.33}$$

we can see that the underlined terms add up to get

$$3\psi : 2\delta L = -i\epsilon\left[\partial_\gamma\partial_\delta g_{\alpha\beta}\dot{\phi}^\gamma\overline{\psi}^\delta\overline{\psi}^\alpha\psi^\beta + \partial_\gamma\partial_\beta g_{\delta\alpha}\dot{\phi}^\alpha\overline{\psi}^\delta\overline{\psi}^\gamma\psi^\beta\right] + \frac{i}{2}\epsilon\left[\left(2\partial_J\partial_K g_{IL} + \underline{\partial_I\partial_L g_{JK}}\right)\dot{\phi}^I\overline{\psi}^L\overline{\psi}^J\psi^K + \underline{\partial_J\partial_K g_{IL}\dot{\phi}^K\overline{\psi}^J\overline{\psi}^L\psi^I}\right]. \tag{3.34}$$

Similarly, the two underlined terms add up to get

$$3\psi : 2\delta L = -i\epsilon \left[\partial_\gamma \partial_\delta g_{\alpha\beta} \dot{\phi}^\gamma \overline{\psi}^\delta \overline{\psi}^\alpha \psi^\beta + \partial_\gamma \partial_\beta g_{\delta\alpha} \dot{\phi}^\alpha \overline{\psi}^\delta \overline{\psi}^\gamma \psi^\beta\right] \tag{3.35}$$
$$+ i\epsilon \left(\partial_J \partial_K g_{IL} + \partial_I \partial_L g_{JK}\right) \dot{\phi}^I \overline{\psi}^L \overline{\psi}^J \psi^K = 0,$$

which we can see reduces to zero.

Now moving on to the five fermion part, we can insert the definition of the Riemann curvature tensor to get

$$5\psi : 2\delta L = \frac{\epsilon}{2} \partial_\gamma \left(\partial_J \partial_K g_{IL} + \partial_I \partial_L g_{JK} - \partial_J \partial_L g_{IK} - \partial_I \partial_K g_{JL}\right) \overline{\psi}^\gamma \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L. \tag{3.36}$$

Now since the $\overline{\psi}$ anti-commute while the order of derivatives does not matter, the first three terms are zero.

$$5\psi : 2\delta L = -\frac{\epsilon}{2} \partial_\gamma \partial_I \partial_K g_{JL} \overline{\psi}^\gamma \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L \tag{3.37}$$

Now since the metric is symmetric while $\overline{\psi}^J$ and $\overline{\psi}^L$ anti-commute, the remaining term is also zero, showing the the action is invariant.

So by the Noether procedure, by taking ϵ to be time dependent, we can find the conserved supercharges

$$Q = i g_{IJ} \overline{\psi}^I \dot{\phi}^J \tag{3.38}$$

$$Q^\dagger = -i g_{IJ} \psi^I \dot{\phi}^J. \tag{3.39}$$

To show this, we vary the Lagrangian as we did previously in eq.(3.9) only taking the ϵ terms. However, since ϵ is now time dependent, the only terms that will change are

the terms containing time derivatives.

$$\begin{aligned}2\delta L =&\epsilon g_{IJ}\partial_t\overline{\psi}^I\dot{\phi}^J + \epsilon g_{IJ}\dot{\phi}^I\partial_t\overline{\psi}^J \\ &+ ig_{IJ}\left[\underline{i\overline{\psi}^I\dot{\epsilon}\dot{\phi}^J} + \overline{\psi}^I\epsilon\partial_t\left(i\dot{\phi}^J - \Gamma^J_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\right) + \overline{\psi}^I\partial_\gamma\Gamma^J_{\alpha\beta}\epsilon\overline{\psi}^\gamma\dot{\phi}^\alpha\psi^\beta\right. \\ &\left.-\epsilon\partial_\gamma\Gamma^I_{\alpha\beta}\overline{\psi}^\gamma\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J - i\partial_t\overline{\psi}^I\epsilon\dot{\phi}^J\right] + \partial_\gamma R_{IJKL}\epsilon\overline{\psi}^\gamma\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L \\ &+ R_{IJKL}\left[i\epsilon\dot{\phi}^I\overline{\psi}^J\psi^K\overline{\psi}^L + i\epsilon\dot{\phi}^K\psi^I\overline{\psi}^J\overline{\psi}^L\right].\end{aligned} \tag{3.40}$$

Comparing eq.(3.40) to eq.(3.10), we can see that there is only one new term. So taking the time integral of the variation of the Lagrangian, the original terms still go to zero while we have

$$2\delta\int L dt = i\int ig_{IJ}\overline{\psi}^I\dot{\epsilon}\dot{\phi}^J dt. \tag{3.41}$$

Now pulling the $\dot{\epsilon}$ out front, we pick up a negative sign.

$$2\delta\int L dt = -i\int\dot{\epsilon}\left(ig_{IJ}\overline{\psi}^I\dot{\phi}^J\right)dt. \tag{3.42}$$

From here we can see that the conserved quantity is the supercharge $Q = ig_{IJ}\overline{\psi}^I\dot{\phi}^J$. To get the other supercharge, the same process can be done for the $\bar{\epsilon}$ part or simply take the complex conjugate of Q to get $Q^\dagger = -ig_{IJ}\psi^I\dot{\phi}^J$.

Also note that the Lagrangian is also invariant under the phase rotation of the fermions

$$\psi^I \to e^{-i\gamma}\psi^I, \overline{\psi}^I \to e^{i\gamma}\overline{\psi}^I; \tag{3.43}$$

where γ is a constant. This can be easily shown by make the substitutions and noting each term has both of the phase rotations reduce each other. The corresponding Noether charge for the transformation is

$$F = g_{IJ}\overline{\psi}^I\psi^J. \tag{3.44}$$

To show this, by the Noether procedure we take γ to be time dependent. Doing

this will only affect the terms containing time derivatives. So looking at the covariant time derivative terms under this transformation

$$\frac{i}{2}g_{IJ}\Big[e^{i\gamma}\overline{\psi}^I D_t\left(e^{-i\gamma}\psi^J\right) - D_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J\Big] = \frac{i}{2}g_{IJ}\Big[e^{i\gamma}\overline{\psi}^I\partial_t\left(e^{-i\gamma}\psi^J\right) + e^{i\gamma}\overline{\psi}^I\Gamma^J_{\alpha\beta}\dot{\phi}^\alpha e^{-i\gamma}\psi^\beta - \partial_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J - \Gamma^I_{\alpha\beta}\dot{\phi}^\alpha e^{i\gamma}\overline{\psi}^\beta e^{-i\gamma}\psi^J\Big], \tag{3.45}$$

we can see that most of the phase rotations reduce each other immediately.

$$\frac{i}{2}g_{IJ}[e^{i\gamma}\overline{\psi}^I D_t\left(e^{-i\gamma}\psi^J\right) - D_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J] = \frac{i}{2}g_{IJ}\left[e^{i\gamma}\overline{\psi}^I\partial_t\left(e^{-i\gamma}\psi^J\right) + \overline{\psi}^I\Gamma^J_{\alpha\beta}\dot{\phi}^\alpha\psi^\beta - \partial_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J - \Gamma^I_{\alpha\beta}\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J\right]. \tag{3.46}$$

Then evaluating the time derivatives, we get

$$\frac{i}{2}g_{IJ}[e^{i\gamma}\overline{\psi}^I D_t\left(e^{-i\gamma}\psi^J\right) - D_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J] = \frac{i}{2}g_{IJ}\Big[-i\dot{\gamma}e^{i\gamma}\overline{\psi}^I e^{-i\gamma}\psi^J + e^{i\gamma}\overline{\psi}^I e^{-i\gamma}\partial_t\psi^J + \overline{\psi}^I\Gamma^J_{\alpha\beta}\dot{\phi}^\alpha\psi^\beta - i\dot{\gamma}e^{i\gamma}\overline{\psi}^I e^{-i\gamma}\psi^J - e^{i\gamma}\partial_t\overline{\psi}^I e^{-i\gamma}\psi^J - \Gamma^I_{\alpha\beta}\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J\Big]. \tag{3.47}$$

Now reducing the remaining phase rotations yields

$$\frac{i}{2}g_{IJ}[e^{i\gamma}\overline{\psi}^I D_t\left(e^{-i\gamma}\psi^J\right) - D_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J] = \frac{i}{2}g_{IJ}\Big[-i\dot{\gamma}\overline{\psi}^I\psi^J + \overline{\psi}^I\partial_t\psi^J + \overline{\psi}^I\Gamma^J_{\alpha\beta}\dot{\phi}^\alpha\psi^\beta - i\dot{\gamma}\overline{\psi}^I\psi^J - \partial_t\overline{\psi}^I\psi^J - \Gamma^I_{\alpha\beta}\dot{\phi}^\alpha\overline{\psi}^\beta\psi^J\Big]. \tag{3.48}$$

Here we can rewrite this in terms of covariant time derivatives as

$$\frac{i}{2}g_{IJ}\left[e^{i\gamma}\overline{\psi}^I D_t\left(e^{-i\gamma}\psi^J\right) - D_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J\right] = \frac{i}{2}g_{IJ}\Big[-2i\dot{\gamma}\overline{\psi}^I\psi^J + \overline{\psi}^I D_t\psi^J - D_t\overline{\psi}^I\psi^J\Big]. \tag{3.49}$$

Separating the $\dot{\gamma}$ terms out gives us

$$\frac{i}{2}g_{IJ}\left[e^{i\gamma}\overline{\psi}^I D_t\left(e^{-i\gamma}\psi^J\right) - D_t\left(e^{i\gamma}\overline{\psi}^I\right)e^{-i\gamma}\psi^J\right] = \dot{\gamma}g_{IJ}\overline{\psi}^I\psi^J + \frac{i}{2}g_{IJ}\left[\overline{\psi}^I D_t\psi^J - D_t\overline{\psi}^I\psi^J\right]. \tag{3.50}$$

So under this transformation, we have

$$L+\delta L = \frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J + \frac{i}{2}g_{IJ}\left(\overline{\psi}^I D_t\psi^J - D_t\overline{\psi}^I\psi^J\right) + \frac{1}{2}R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L + \dot{\gamma}g_{IJ}\overline{\psi}^I\psi^J. \tag{3.51}$$

This means that

$$\delta\int L dt = \int \dot{\gamma}g_{IJ}\overline{\psi}^I\psi^J dt \tag{3.52}$$

and the corresponding Noether charge is $F = g_{IJ}\overline{\psi}^I\psi^J$.

3.2 Special Case: S^2

For example working on S^2, the metric tensor and inverse are

$$g_{IJ} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\phi^1 \end{pmatrix} \quad g^{IJ} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\phi^1} \end{pmatrix}, \tag{3.53}$$

where $\{r, \phi^1, \phi^2\}$ are $\{r, \theta, \phi\}$ from spherical coordinates respectively. Using the definition for the connection coefficients, the nonzero connections for S^2 are

$$\Gamma^1_{22} = -\sin\phi^1\cos\phi^1 \tag{3.54}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \cot\phi^1. \tag{3.55}$$

Here we define the covariant derivatives as

$$D_J v^I = \partial_J v^I + \Gamma^I_{JK} v^K \tag{3.56}$$

$$D_t \psi^I = \partial_t \psi^I + \Gamma^I_{JK} \dot{\phi}^J \psi^K. \tag{3.57}$$

Using the definition for the Riemann curvature tensor

$$R_{IJKL} = g_{IM} R^M_{JKL} = g_{IM} \left[\partial_K \Gamma^M_{JL} - \partial_L \Gamma^M_{JK} + \Gamma^M_{KN} \Gamma^N_{JL} - \Gamma^M_{LN} \Gamma^N_{JK} \right] \tag{3.58}$$

the nonzero components of the tensor are

$$R_{12KL} = -R_{21KL} = \begin{pmatrix} 0 & r^2 \sin^2 \phi^1 \\ -r^2 \sin^2 \phi^1 & 0 \end{pmatrix}. \tag{3.59}$$

Now using

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{i}{2} g_{IJ} \left(\overline{\psi}^I D_t \psi^J - D_t \overline{\psi}^I \psi^J \right) + \frac{1}{2} R_{IJKL} \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L \tag{3.60}$$

as our Lagrangian on S^2, expanding everything out gives us

$$\begin{aligned} L =& \frac{1}{2} r^2 \left[\left(\dot{\phi}^1 \right)^2 + \sin^2 \phi^1 \left(\dot{\phi}^2 \right)^2 \right] \\ &+ \frac{i}{2} r^2 \left[\overline{\psi}^1 \left(\partial_t \psi^1 - \sin \phi^1 \cos \phi^1 \dot{\phi}^2 \psi^2 \right) - \left(\partial_t \overline{\psi}^1 - \sin \phi^1 \cos \phi^1 \dot{\phi}^2 \overline{\psi}^2 \right) \psi^1 \right] \\ &- 2 \sin^2 \left(\phi^1 \right) \psi^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1. \end{aligned} \tag{3.61}$$

Note that due to the symmetry of the last two components of the Riemann curvature tensor and the antisymmetry of the the fermionic variables, $R_{1212} = 0 = R_{2121}$. Now for convenience, we can take all of the common factors from the right hand side to

the left.

$$\begin{aligned}\frac{2L}{r^2} &= \left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 \\ &+ i\left[\overline{\psi}^1\left(\partial_t\psi^1 - \sin\phi^1\cos\phi^1\dot{\phi}^2\psi^2\right) - \left(\partial_t\overline{\psi}^1 - \sin\phi^1\cos\phi^1\dot{\phi}^2\overline{\psi}^2\right)\psi^1\right] \\ &- 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.62}$$

Due to the symmetries the Riemann curvature tensor, the last two terms can be combined together.

$$\begin{aligned}\frac{2L}{r^2} &= \left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 \\ &+ i\left[\overline{\psi}^1\left(\partial_t\psi^1 - \sin\phi^1\cos\phi^1\dot{\phi}^2\psi^2\right) - \left(\partial_t\overline{\psi}^1 - \sin\phi^1\cos\phi^1\dot{\phi}^2\overline{\psi}^2\right)\psi^1\right] \\ &- 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1\end{aligned} \tag{3.63}$$

Now distributing the $\sin^2\phi^1$, we can regroup the terms as

$$\begin{aligned}\frac{2L}{r^2} &= \left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 \\ &+ i\left[\overline{\psi}^1\left(\partial_t\psi^1 - \sin\phi^1\cos\phi^1\dot{\phi}^2\psi^2\right) - \left(\partial_t\overline{\psi}^1 - \sin\phi^1\cos\phi^1\dot{\phi}^2\overline{\psi}^2\right)\psi^1\right. \\ &+ \overline{\psi}^2\left(\sin^2\phi^1\partial_t\psi^2 + \sin\phi^1\cos\phi^1\dot{\phi}^1\psi^2 + \sin\phi^1\cos\phi^1\dot{\phi}^2\psi^1\right) \\ &\left. - \left(\sin^2\phi^1\partial_t\overline{\psi}^2 + \sin\phi^1\cos\phi^1\dot{\phi}^1\overline{\psi}^2 + \sin\phi^1\cos\phi^1\dot{\phi}^2\overline{\psi}^1\right)\psi^2\right] \\ &- 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.64}$$

Here combining like terms gives us

$$\begin{aligned}\frac{2L}{r^2} &= \left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 \\ &+ i\left[\overline{\psi}^1\partial_t\psi^1 - \partial_t\overline{\psi}^1\psi^1 + \overline{\psi}^2\left(\sin^2\phi^1\partial_t\psi^2 + 2\sin\phi^1\cos\phi^1\dot{\phi}^2\psi^1\right)\right. \\ &\left. - \left(\sin^2\phi^1\partial_t\overline{\psi}^2 + 2\sin\phi^1\cos\phi^1\dot{\phi}^2\overline{\psi}^1\right)\psi^2\right] - 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.65}$$

Now using the trigonometric identity $2\sin\phi^1\cos\phi^1 = \sin(2\phi^1)$, we have

$$\begin{aligned}\frac{2L}{r^2} &= \left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 \\ &+ i\left[\overline{\psi}^1\partial_t\psi^1 - \partial_t\overline{\psi}^1\psi^1 + \overline{\psi}^2\left(\sin^2\phi^1\partial_t\psi^2 + \sin\left(2\phi^1\right)\dot{\phi}^2\psi^1\right)\right. \\ &\left. - \left(\sin^2\phi^1\partial_t\overline{\psi}^2 + \sin\left(2\phi^1\right)\dot{\phi}^2\overline{\psi}^1\right)\psi^2\right] - 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.66}$$

Then grouping the imaginary terms by their leading trigonometric functions, we have

$$\begin{aligned}\frac{2L}{r^2} &= \left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 \\ &+ i\left[\overline{\psi}^1\partial_t\psi^1 - \partial_t\overline{\psi}^1\psi^1 + \sin\left(2\phi^1\right)\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right. \\ &\left. + \sin^2\phi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] - 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.67}$$

Now taking the variation of the Lagrangian gives us

$$\begin{aligned}\frac{2\delta L}{r^2} &= \underline{2\dot{\phi}^1\delta\dot{\phi}^1} + \sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2\delta\phi^1 + \underline{2\sin^2\phi^1\dot{\phi}^2\delta\dot{\phi}^2} + i\left[\delta\overline{\psi}^1\partial_t\psi^1 + \underline{\overline{\psi}^1\partial_t\delta\psi^1}\right. \\ &\left. - \underline{\partial_t\delta\overline{\psi}^1\psi^1} - \partial_t\overline{\psi}^1\delta\psi^1 + 2\cos\left(2\phi^1\right)\delta\phi^1\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right. \\ &+ \sin\left(2\phi^1\right)\left(\delta\dot{\phi}^2\overline{\psi}^2\psi^1 + \dot{\phi}^2\delta\overline{\psi}^2\psi^1 + \dot{\phi}^2\overline{\psi}^2\delta\psi^1 - \delta\dot{\phi}^2\overline{\psi}^1\psi^2 - \dot{\phi}^2\delta\overline{\psi}^1\psi^2 - \dot{\phi}^2\overline{\psi}^1\delta\psi^2\right) \\ &+ \sin\left(2\phi^1\right)\delta\phi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right) \\ &\left. + \sin^2\phi^1\left(\delta\overline{\psi}^2\partial_t\psi^2 + \overline{\psi}^2\partial_t\delta\psi^2 - \partial_t\delta\overline{\psi}^2\psi^2 - \partial_t\overline{\psi}^2\delta\psi^2\right)\right].\end{aligned} \tag{3.68}$$

Using integration by parts on the underlined terms, we can combine like terms to simplify

$$\begin{aligned}\frac{2\delta L}{r^2} &= -2\ddot{\phi}^1\delta\phi^1 + \sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2\delta\phi^1 - \partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\delta\phi^2 + i\left[2\delta\overline{\psi}^1\partial_t\psi^1\right. \\ &- 2\partial_t\overline{\psi}^1\delta\psi^1 + 2\cos\left(2\phi^1\right)\delta\phi^1\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right) - \delta\phi^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right] \\ &+ \sin\left(2\phi^1\right)\left[\dot{\phi}^2\delta\overline{\psi}^2\psi^1 + \dot{\phi}^2\overline{\psi}^2\delta\psi^1 - \dot{\phi}^2\delta\overline{\psi}^1\psi^2 - \dot{\phi}^2\overline{\psi}^1\delta\psi^2 + \delta\phi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] \\ &\left. + \sin^2\phi^1\left(\delta\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\delta\psi^2\right) - \partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\delta\psi^2 + \delta\overline{\psi}^2\partial_t\left(\sin^2\phi^1\psi^2\right)\right].\end{aligned} \tag{3.69}$$

Using the supersymmetry relations

$$\begin{aligned}
\delta\phi^1 &= \epsilon\overline{\psi}^1 - \bar{\epsilon}\psi^1 \\
\delta\phi^2 &= \epsilon\overline{\psi}^2 - \bar{\epsilon}\psi^2 \\
\delta\psi^1 &= \epsilon\left(i\dot{\phi}^1 + \sin\phi^1\cos\phi^1\overline{\psi}^2\psi^2\right) \\
\delta\psi^2 &= \epsilon\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right) \\
\delta\overline{\psi}^1 &= \bar{\epsilon}\left(-i\dot{\phi}^1 + \sin\phi^1\cos\phi^1\overline{\psi}^2\psi^2\right) \\
\delta\overline{\psi}^2 &= \bar{\epsilon}\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)
\end{aligned} \tag{3.70}$$

gives us

$$\begin{aligned}
\frac{2\delta L}{r^2} &= \left(-2\ddot{\phi}^1 + \sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2\right)\left(\epsilon\overline{\psi}^1 - \bar{\epsilon}\psi^1\right) - \partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\left(\epsilon\overline{\psi}^2 - \bar{\epsilon}\psi^2\right) \\
&+ i\left[2\bar{\epsilon}\left(-i\dot{\phi}^1 + \sin\phi^1\cos\phi^1\overline{\psi}^2\psi^2\right)\partial_t\psi^1 - 2\partial_t\overline{\psi}^1\epsilon\left(i\dot{\phi}^1 + \sin\phi^1\cos\phi^1\overline{\psi}^2\psi^2\right)\right. \\
&\quad + 2\cos\left(2\phi^1\right)\left(\epsilon\overline{\psi}^1 - \bar{\epsilon}\psi^1\right)\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right) \\
&\quad - \left(\epsilon\overline{\psi}^2 - \bar{\epsilon}\psi^2\right)\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right] \\
&\quad + \sin\left(2\phi^1\right)\left[\dot{\phi}^2\bar{\epsilon}\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2\right)\psi^1 + \dot{\phi}^2\overline{\psi}^2\epsilon\left(i\dot{\phi}^1\right) - \dot{\phi}^2\bar{\epsilon}\left(-i\dot{\phi}^1\right)\psi^2\right. \\
&\qquad \left.-\dot{\phi}^2\overline{\psi}^1\epsilon\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^2\psi^1\right) + \left(\epsilon\overline{\psi}^1 - \bar{\epsilon}\psi^1\right)\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] \\
&\quad + \sin^2\phi^1\left[\bar{\epsilon}\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2\right. \\
&\qquad \left.-\partial_t\overline{\psi}^2\epsilon\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\right] \\
&\quad - \partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\epsilon\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad \left.+\bar{\epsilon}\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\right].
\end{aligned} \tag{3.71}$$

Next separating all of the terms by ϵ and $\bar{\epsilon}$ will help us with the mathematical bookkeeping.

$$
\begin{aligned}
\frac{2\delta L}{r^2} = \epsilon \Big\{ & \left(\underline{-2\ddot{\phi}^1} + \sin\left(2\phi^1\right) \left(\dot{\phi}^2\right)^2 \right) \overline{\psi}^1 - \partial_t \left(2\sin^2\phi^1 \dot{\phi}^2 \right) \overline{\psi}^2 \\
& + i \Big[2\partial_t \overline{\psi}^1 \left(\underline{i\dot{\phi}^1} + \sin\phi^1 \cos\phi^1 \overline{\psi}^2 \psi^2 \right) + 2\cos\left(2\phi^1\right) \overline{\psi}^1 \dot{\phi}^2 \overline{\psi}^2 \psi^1 \\
& - \overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] + \sin\left(2\phi^1\right) \Big[-i\dot{\phi}^1 \dot{\phi}^2 \overline{\psi}^2 \\
& + \dot{\phi}^2 \overline{\psi}^1 \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) + \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \Big] \\
& + \sin^2\phi^1 \partial_t \overline{\psi}^2 \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \\
& + \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \Big] \\
& - 2i \sin^2\phi^1 \left(\dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 + \dot{\phi}^2 \psi^1 \overline{\psi}^2 \overline{\psi}^1 \right) \Big\} \\
& + \bar{\epsilon} \Big\{ \left(\underline{2\ddot{\phi}^1} - \sin\left(2\phi^1\right) \left(\dot{\phi}^2\right)^2 \right) \psi^1 + \partial_t \left(2\sin^2\phi^1 \dot{\phi}^2 \right) \psi^2 \\
& + i \Big[2 \left(\underline{-i\dot{\phi}^1} + \sin\phi^1 \cos\phi^1 \overline{\psi}^2 \psi^2 \right) \partial_t \psi^1 + 2\cos\left(2\phi^1\right) \psi^1 \dot{\phi}^2 \overline{\psi}^1 \psi^2 \\
& + \psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \left[\dot{\phi}^2 \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 \right) \psi^1 + i\dot{\phi}^1 \dot{\phi}^2 \psi^2 - \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \right] \\
& + \sin^2\phi^1 \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& + \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) \Big] \\
& - 2i \sin^2\phi^1 \left(\dot{\phi}^2 \psi^1 \psi^2 \overline{\psi}^1 + \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \right) \Big\} .
\end{aligned}
\tag{3.72}
$$

By noting that the underlined terms are a total time derivative, they can be dropped from the Lagrangian without affecting the variation of the action. Also using the trigonometric identity $\sin\left(2\phi^1\right) = 2\sin\phi^1 \cos\phi^1$, we have

$$
\begin{aligned}
\frac{2\delta L}{r^2} = \epsilon \Big\{ & \underline{\sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2 \overline{\psi}^1} - \partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\overline{\psi}^2 + i\Big[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 \\
& + 2\cos\left(2\phi^1\right)\overline{\psi}^1\dot{\phi}^2\overline{\psi}^2\psi^1 - \overline{\psi}^2\partial_t\Big[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\Big] \\
& + \sin\left(2\phi^1\right)\Big[-i\dot{\phi}^1\dot{\phi}^2\overline{\psi}^2 + \dot{\phi}^2\overline{\psi}^1\left(\underline{i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^2\psi^1\right) + \overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\Big] \\
& + \sin^2\phi^1\partial_t\overline{\psi}^2\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right) \\
& + \partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\Big] \\
& - 2i\sin^2\phi^1\left(\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1 + \dot{\phi}^2\psi^1\overline{\psi}^2\overline{\psi}^1\right)\Big\} \\
+ \bar{\epsilon}\Big\{ & \underline{-\sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2\psi^1} + \partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\psi^2 + i\Big[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 \\
& + 2\cos\left(2\phi^1\right)\psi^1\dot{\phi}^2\overline{\psi}^1\psi^2 + \psi^2\partial_t\Big[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\Big] \\
& + \sin\left(2\phi^1\right)\Big[\dot{\phi}^2\left(\underline{-i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^1\psi^2\right)\psi^1 + i\dot{\phi}^1\dot{\phi}^2\psi^2 - \psi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\Big] \\
& + \sin^2\phi^1\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
& - \left(i\dot{\phi}^2 + \cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\Big] \\
& - 2i\sin^2\phi^1\left(\dot{\phi}^2\psi^1\psi^2\overline{\psi}^1 + \dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right)\Big\}.
\end{aligned}
\tag{3.73}
$$

Above the underlined terms cancel each other

$$
\begin{aligned}
\frac{2\delta L}{r^2} = \epsilon \Big\{ & \underline{-\partial_t \left(2\sin^2\phi^1 \dot{\phi}^2\right) \overline{\psi}^2} + i \Big[\sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 \\
& + 2\cos\left(2\phi^1\right) \overline{\psi}^1 \dot{\phi}^2 \overline{\psi}^2 \psi^1 - \overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2\right)\right] \\
& + \sin\left(2\phi^1\right) \left[-i\dot{\phi}^1 \dot{\phi}^2 \overline{\psi}^2 - \dot{\phi}^2 \overline{\psi}^1 \cot\phi^1 \overline{\psi}^2 \psi^1 + \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2\right)\right] \\
& + \sin^2\phi^1 \partial_t \overline{\psi}^2 \left(\underline{i\dot{\phi}^2} - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1\right) \\
& + \partial_t \left(\sin^2\phi^1 \overline{\psi}^2\right) \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1\right)\Big] \\
& - 2i\sin^2\phi^1 \left(\dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 + \dot{\phi}^2 \psi^1 \overline{\psi}^2 \overline{\psi}^1\right)\Big\} \\
+ \bar{\epsilon} \Big\{ & \underline{\partial_t \left(2\sin^2\phi^1 \dot{\phi}^2\right) \psi^2} + i \Big[\sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 \\
& + 2\cos\left(2\phi^1\right) \psi^1 \dot{\phi}^2 \overline{\psi}^1 \psi^2 + \psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2\right)\right] \\
& + \sin\left(2\phi^1\right) \left[-\dot{\phi}^2 \cot\phi^1 \overline{\psi}^1 \psi^2 \psi^1 + i\dot{\phi}^1 \dot{\phi}^2 \psi^2 - \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2\right)\right] \\
& + \sin^2\phi^1 \left(\underline{-i\dot{\phi}^2} - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1\right) \partial_t \psi^2 \\
& + \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1\right) \partial_t \left(\sin^2\phi^1 \psi^2\right)\Big] \\
& - 2i\sin^2\phi^1 \left(\dot{\phi}^2 \psi^1 \psi^2 \overline{\psi}^1 + \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2\right)\Big\}.
\end{aligned}
\tag{3.74}
$$

By using integration by parts on the $\pm i\dot{\phi}^2$ terms to isolate $\overline{\psi}^2$, we can cancel it with one of partial time derivatives

$$
\begin{aligned}
\frac{2\delta L}{r^2} = \epsilon \Big\{ & \underline{-\partial_t \left(\sin^2\phi^1 \dot{\phi}^2\right) \overline{\psi}^2} + i \Big[\sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 \\
& + 2\cos\left(2\phi^1\right) \overline{\psi}^1 \dot{\phi}^2 \overline{\psi}^2 \psi^1 - \overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \left[\underline{-i\dot{\phi}^1 \dot{\phi}^2 \overline{\psi}^2} - \dot{\phi}^2 \overline{\psi}^1 \cot\phi^1 \overline{\psi}^2 \psi^1 + \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \right] \\
& - \sin^2\phi^1 \cot\phi^1 \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \\
& + \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(\underline{i\dot{\phi}^2} - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \Big] \\
& - 2i \sin^2\phi^1 \left(\dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 + \dot{\phi}^2 \psi^1 \overline{\psi}^2 \overline{\psi}^1 \right) \Big\} \\
+ \overline{\epsilon} \Big\{ & \underline{\partial_t \left(2\sin^2\phi^1 \dot{\phi}^2 \right) \psi^2} + i \Big[\sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 \\
& + 2\cos\left(2\phi^1\right) \psi^1 \dot{\phi}^2 \overline{\psi}^1 \psi^2 + \psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \left[-\dot{\phi}^2 \cot\phi^1 \overline{\psi}^1 \psi^2 \psi^1 + i\dot{\phi}^1 \dot{\phi}^2 \psi^2 - \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \right] \\
& + \sin^2\phi^1 \left(\underline{-i\dot{\phi}^2} - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& + \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) \Big] \\
& - 2i \sin^2\phi^1 \left(\dot{\phi}^2 \psi^1 \psi^2 \overline{\psi}^1 + \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \right) \Big\} .
\end{aligned}
\tag{3.75}
$$

Again using integration by parts on the $\underline{\pm i\dot{\phi}^2}$ terms will cancel with the remaining underlined terms above

$$
\begin{aligned}
\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \underline{2\cos\left(2\phi^1\right) \overline{\psi}^1 \dot{\phi}^2 \overline{\psi}^2 \psi^1} \\
& - \overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \left[-\underline{\dot{\phi}^2 \overline{\psi}^1 \cot\phi^1 \overline{\psi}^2 \psi^1} + \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \right] \\
& - \sin^2\phi^1 \cot\phi^1 \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) - \cot\phi^1 \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \\
& - 2\sin^2\phi^1 \left(\dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 + \underline{\dot{\phi}^2 \psi^1 \overline{\psi}^2 \overline{\psi}^1} \right) \Big\} \\
+ i\bar{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \underline{2\cos\left(2\phi^1\right) \psi^1 \dot{\phi}^2 \overline{\psi}^1 \psi^2} \\
& + \psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \left[-\underline{\dot{\phi}^2 \cot\phi^1 \overline{\psi}^1 \psi^2 \psi^1} - \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \right] \\
& - \sin^2\phi^1 \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 - \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) \\
& - 2\sin^2\phi^1 \left(\underline{\dot{\phi}^2 \psi^1 \psi^2 \overline{\psi}^1} + \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \right) \Big\} .
\end{aligned}
\tag{3.76}
$$

Noting the $\dot{\phi}^2$ terms, we can use trigonometric identities and some rearranging of the fermionic variables

$$
\begin{aligned}
\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \underline{2\left(\cos^2\phi^1 - \sin^2\phi^1\right) \overline{\psi}^1 \dot{\phi}^2 \overline{\psi}^2 \psi^1} \\
& - \overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] - \underline{2\cos^2\phi^1 \dot{\phi}^2 \overline{\psi}^1 \overline{\psi}^2 \psi^1} \\
& + \sin\left(2\phi^1\right) \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) - \sin^2\phi^1 \cot\phi^1 \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \\
& - \cot\phi^1 \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) - 2\sin^2\phi^1 \left(\dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 - \underline{\dot{\phi}^2 \overline{\psi}^1 \overline{\psi}^2 \psi^1} \right) \Big\} \\
+ i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \underline{2\left(\cos^2\phi^1 - \sin^2\phi^1\right) \dot{\phi}^2 \psi^1 \overline{\psi}^1 \psi^2} \\
& + \psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] - \underline{2\cos^2\phi^1 \dot{\phi}^2 \psi^1 \overline{\psi}^1 \psi^2} \\
& - \sin\left(2\phi^1\right) \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) - \sin^2\phi^1 \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& - \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) - 2\sin^2\phi^1 \left(\underline{-\dot{\phi}^2 \psi^1 \overline{\psi}^1 \psi^2} + \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \right) \Big\},
\end{aligned}
\tag{3.77}
$$

so that they can cancel each other.

$$
\begin{aligned}
\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 - \underline{\overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right]} \\
& + \sin\left(2\phi^1\right) \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) - \sin^2\phi^1 \cot\phi^1 \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \\
& - \cot\phi^1 \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) - 2\sin^2\phi^1 \dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 \Big\} \\
+ i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \underline{\psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right]} \\
& - \sin\left(2\phi^1\right) \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) - \sin^2\phi^1 \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& - \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) - 2\sin^2\phi^1 \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \Big\}
\end{aligned}
\tag{3.78}
$$

Then using integration by parts on the underlined terms gives us

$$
\begin{aligned}
\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \partial_t \overline{\psi}^2 \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \underline{\overline{\psi}^1 \psi^2} \right) \right] \\
& + \sin\left(2\phi^1\right) \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \underline{\partial_t \overline{\psi}^2 \psi^2} \right) - \sin^2\phi^1 \cot\phi^1 \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \\
& - \cot\phi^1 \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) - 2\sin^2\phi^1 \dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 \Big\} \\
+ i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 - \partial_t \psi^2 \left[\sin\left(2\phi^1\right) \left(\underline{\overline{\psi}^2 \psi^1} - \overline{\psi}^1 \psi^2 \right) \right] \\
& - \sin\left(2\phi^1\right) \psi^1 \left(\underline{\overline{\psi}^2 \partial_t \psi^2} - \partial_t \overline{\psi}^2 \psi^2 \right) - \sin^2\phi^1 \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& - \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) - 2\sin^2\phi^1 \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \Big\} .
\end{aligned}
\tag{3.79}
$$

Then the underlined terms cancel each other giving us

$$
\begin{aligned}
\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \sin\left(2\phi^1\right) \partial_t \overline{\psi}^2 \overline{\psi}^2 \psi^1 \\
& + \sin\left(2\phi^1\right) \overline{\psi}^1 \overline{\psi}^2 \partial_t \psi^2 - \sin^2\phi^1 \cot\phi^1 \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \\
& - \cot\phi^1 \underline{\partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right)} \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) - 2\sin^2\phi^1 \dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 \Big\} \\
+ i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \sin\left(2\phi^1\right) \partial_t \psi^2 \overline{\psi}^1 \psi^2 \\
& + \sin\left(2\phi^1\right) \psi^1 \partial_t \overline{\psi}^2 \psi^2 - \sin^2\phi^1 \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& - \cot\phi^1 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1 \right) \underline{\partial_t \left(\sin^2\phi^1 \psi^2 \right)} - 2\sin^2\phi^1 \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \Big\} .
\end{aligned}
\tag{3.80}
$$

Evaluating the time derivatives gives us

$$\begin{aligned}\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \sin\left(2\phi^1\right) \partial_t \overline{\psi}^2 \overline{\psi}^2 \psi^1 \\ & + \sin\left(2\phi^1\right) \overline{\psi}^1 \overline{\psi}^2 \partial_t \psi^2 - \underline{2 \sin^2\phi^1 \cot\phi^1} \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1\right) \\ & - \cot\phi^1 \sin\left(2\phi^1\right) \dot{\phi}^1 \overline{\psi}^2 \overline{\psi}^1 \psi^2 - 2 \sin^2\phi^1 \dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 \Big\} \\ + i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \sin\left(2\phi^1\right) \partial_t \psi^2 \overline{\psi}^1 \psi^2 \\ & + \sin\left(2\phi^1\right) \psi^1 \partial_t \overline{\psi}^2 \psi^2 - \underline{2 \sin^2\phi^1 \cot\phi^1} \left(\overline{\psi}^1 \psi^2 + \overline{\psi}^2 \psi^1\right) \partial_t \psi^2 \\ & - \cot\phi^1 \sin\left(2\phi^1\right) \dot{\phi}^1 \overline{\psi}^2 \psi^1 \psi^2 - 2 \sin^2\phi^1 \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \Big\} . \end{aligned} \tag{3.81}$$

Using trigonometric identities here simplify the underlined terms to get

$$\begin{aligned}\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \underline{\sin\left(2\phi^1\right) \partial_t \overline{\psi}^2 \overline{\psi}^2 \psi^1} \\ & + \sin\left(2\phi^1\right) \overline{\psi}^1 \overline{\psi}^2 \partial_t \psi^2 - \sin\left(2\phi^1\right) \partial_t \overline{\psi}^2 \left(\overline{\psi}^1 \psi^2 + \underline{\overline{\psi}^2 \psi^1}\right) \\ & - \cot\phi^1 \sin\left(2\phi^1\right) \dot{\phi}^1 \overline{\psi}^2 \overline{\psi}^1 \psi^2 - 2 \sin^2\phi^1 \dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 \Big\} \\ + i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \underline{\sin\left(2\phi^1\right) \partial_t \psi^2 \overline{\psi}^1 \psi^2} \\ & + \sin\left(2\phi^1\right) \psi^1 \partial_t \overline{\psi}^2 \psi^2 - \sin\left(2\phi^1\right) \left(\underline{\overline{\psi}^1 \psi^2} + \overline{\psi}^2 \psi^1\right) \partial_t \psi^2 \\ & - \cot\phi^1 \sin\left(2\phi^1\right) \dot{\phi}^1 \overline{\psi}^2 \psi^1 \psi^2 - 2 \sin^2\phi^1 \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \Big\} . \end{aligned} \tag{3.82}$$

Here the underlined terms cancel each other to get

$$\begin{aligned}\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right) \partial_t \overline{\psi}^1 \overline{\psi}^2 \psi^2 + \sin\left(2\phi^1\right) \overline{\psi}^1 \overline{\psi}^2 \partial_t \psi^2 - \sin\left(2\phi^1\right) \partial_t \overline{\psi}^2 \overline{\psi}^1 \psi^2 \\ & - \underline{\cot\phi^1 \sin\left(2\phi^1\right)} \dot{\phi}^1 \overline{\psi}^2 \overline{\psi}^1 \psi^2 - 2 \sin^2\phi^1 \dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 \Big\} \\ + i\overline{\epsilon} \Big\{ & \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^2 \partial_t \psi^1 + \sin\left(2\phi^1\right) \psi^1 \partial_t \overline{\psi}^2 \psi^2 - \sin\left(2\phi^1\right) \overline{\psi}^2 \psi^1 \partial_t \psi^2 \\ & - \underline{\cot\phi^1 \sin\left(2\phi^1\right)} \dot{\phi}^1 \overline{\psi}^2 \psi^1 \psi^2 - 2 \sin^2\phi^1 \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \Big\} . \end{aligned} \tag{3.83}$$

Simplifying the underlined terms using trigonometric identities gives us

$$\begin{aligned}\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 \sin\left(2\phi^1\right)\partial_t\overline{\psi}^2\overline{\psi}^1\psi^2 \\ & -2\cos^2\phi^1\dot{\phi}^1\overline{\psi}^2\overline{\psi}^1\psi^2 - 2\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\Big\} \\ + i\overline{\epsilon}\Big\{ & \sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 + \sin\left(2\phi^1\right)\psi^1\partial_t\overline{\psi}^2\psi^2 - \sin\left(2\phi^1\right)\overline{\psi}^2\psi^1\partial_t\psi^2 \\ & -2\cos^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^1\psi^2 - 2\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\Big\}.\end{aligned} \tag{3.84}$$

Here we can rearrange the fermionic variables to get

$$\begin{aligned}\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 \\ & + \sin\left(2\phi^1\right)\overline{\psi}^1\partial_t\overline{\psi}^2\psi^2 + 2\underline{\left(\cos^2\phi^1 - \sin^2\phi^1\right)}\dot{\phi}^1\overline{\psi}^1\overline{\psi}^2\psi^2\Big\} \\ + i\overline{\epsilon}\Big\{ & \sin\left(2\phi^1\right)\partial_t\psi^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\psi^1\partial_t\overline{\psi}^2\psi^2 \\ & + \sin\left(2\phi^1\right)\psi^1\overline{\psi}^2\partial_t\psi^2 + 2\underline{\left(\cos^2\phi^1 - \sin^2\phi^1\right)}\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\Big\}.\end{aligned} \tag{3.85}$$

Using the double angle cosine trigonometric identity on the underlined terms, we get

$$\begin{aligned}\frac{2\delta L}{r^2} = i\epsilon \Big\{ & \sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 \\ & + \sin\left(2\phi^1\right)\overline{\psi}^1\partial_t\overline{\psi}^2\psi^2 + 2\cos\left(2\phi^1\right)\dot{\phi}^1\overline{\psi}^1\overline{\psi}^2\psi^2\Big\} \\ + i\overline{\epsilon}\Big\{ & \sin\left(2\phi^1\right)\partial_t\psi^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\psi^1\partial_t\overline{\psi}^2\psi^2 \\ & + \sin\left(2\phi^1\right)\psi^1\overline{\psi}^2\partial_t\psi^2 + 2\cos\left(2\phi^1\right)\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\Big\}.\end{aligned} \tag{3.86}$$

Here we can note that

$$\begin{aligned}\partial_t\left[\sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\psi^2\right] = & \ 2\cos\left(2\phi^1\right)\dot{\phi}^1\overline{\psi}^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 \\ & + \sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 + \sin\left(2\phi^1\right)\overline{\psi}^1\partial_t\overline{\psi}^2\psi^2\end{aligned} \tag{3.87}$$

$$\begin{aligned}\partial_t\left[\sin\left(2\phi^1\right)\psi^1\overline{\psi}^2\psi^2\right] = & \ 2\cos\left(2\phi^1\right)\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\partial_t\psi^1\overline{\psi}^2\psi^2 \\ & + \sin\left(2\phi^1\right)\psi^1\partial_t\overline{\psi}^2\psi^2 + \sin\left(2\phi^1\right)\psi^1\overline{\psi}^2\partial_t\psi^2.\end{aligned} \tag{3.88}$$

Since these are total time derivatives, we have that

$$\delta S = \delta \int L\, dt = 0. \tag{3.89}$$

Following from eq.(3.43), the supercharges are

$$Q = ir^2 \left(\overline{\psi}^1 \dot{\phi}^1 + \sin^2 \phi^1 \overline{\psi}^2 \dot{\phi}^2 \right) \tag{3.90}$$

$$Q^\dagger = -ir^2 \left(\psi^1 \dot{\phi}^1 + \sin^2 \phi^1 \psi^2 \dot{\phi}^2 \right), \tag{3.91}$$

while from eq.(3.45), the Noether charge is

$$F = r^2 \left(\overline{\psi}^1 \psi^1 + \sin^2 \phi^1 \overline{\psi}^2 \psi^2 \right). \tag{3.92}$$

To check this, we take the fully expanded variation of the Lagrangian as

$$
\begin{aligned}
\frac{2\delta L}{r^2} = \epsilon \Bigg\{ & \left(\underline{-2\ddot{\phi}^1} + \sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2 \right) \overline{\psi}^1 - \partial_t \left(2\sin^2\phi^1 \dot{\phi}^2 \right) \overline{\psi}^2 \\
& + i \Big[2\partial_t \overline{\psi}^1 \left(\underline{i\dot{\phi}^1} + \sin\phi^1 \cos\phi^1 \overline{\psi}^2 \psi^2 \right) + 2\cos\left(2\phi^1\right) \overline{\psi}^1 \dot{\phi}^2 \overline{\psi}^2 \psi^1 \\
& - \overline{\psi}^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \Big[-i\dot{\phi}^1 \dot{\phi}^2 \overline{\psi}^2 + \dot{\phi}^2 \overline{\psi}^1 \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) + \overline{\psi}^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \Big] \\
& + \sin^2\phi^1 \partial_t \overline{\psi}^2 \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \\
& + \partial_t \left(\sin^2\phi^1 \overline{\psi}^2 \right) \left(i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \Big] \\
& - 2i\sin^2\phi^1 \left(\dot{\phi}^1 \overline{\psi}^2 \psi^2 \overline{\psi}^1 + \dot{\phi}^2 \psi^1 \overline{\psi}^2 \overline{\psi}^1 \right) \Bigg\} \\
+ \bar{\epsilon} \Bigg\{ & \left(\underline{2\ddot{\phi}^1} - \sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2 \right) \psi^1 + \partial_t \left(2\sin^2\phi^1 \dot{\phi}^2 \right) \psi^2 \\
& + i \Big[2 \left(\underline{-i\dot{\phi}^1} + \sin\phi^1 \cos\phi^1 \overline{\psi}^2 \psi^2 \right) \partial_t \psi^1 + 2\cos\left(2\phi^1\right) \psi^1 \dot{\phi}^2 \overline{\psi}^1 \psi^2 \\
& + \psi^2 \partial_t \left[\sin\left(2\phi^1\right) \left(\overline{\psi}^2 \psi^1 - \overline{\psi}^1 \psi^2 \right) \right] \\
& + \sin\left(2\phi^1\right) \Big[\dot{\phi}^2 \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 \right) \psi^1 + i\dot{\phi}^1 \dot{\phi}^2 \psi^2 - \psi^1 \left(\overline{\psi}^2 \partial_t \psi^2 - \partial_t \overline{\psi}^2 \psi^2 \right) \Big] \\
& + \sin^2\phi^1 \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \partial_t \psi^2 \\
& + \left(-i\dot{\phi}^2 - \cot\phi^1 \overline{\psi}^1 \psi^2 - \cot\phi^1 \overline{\psi}^2 \psi^1 \right) \partial_t \left(\sin^2\phi^1 \psi^2 \right) \Big] \\
& - 2i\sin^2\phi^1 \left(\dot{\phi}^2 \psi^1 \psi^2 \overline{\psi}^1 + \dot{\phi}^1 \psi^1 \overline{\psi}^2 \psi^2 \right) \Bigg\} .
\end{aligned}
\tag{3.93}
$$

By noting that the underlined terms are part total time derivative, we can say

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + \epsilon\left\{\underline{\sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2\overline{\psi}^1} - \partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\overline{\psi}^2\right. \\
&\quad + i\left[2\partial_t\overline{\psi}^1\sin\phi^1\cos\phi^1\overline{\psi}^2\psi^2 + 2\cos\left(2\phi^1\right)\overline{\psi}^1\dot{\phi}^2\overline{\psi}^2\psi^1\right. \\
&\quad - \overline{\psi}^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right] \\
&\quad + \sin\left(2\phi^1\right)\left[-i\dot{\phi}^1\dot{\phi}^2\overline{\psi}^2 + \dot{\phi}^2\overline{\psi}^1\left(\underline{i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^2\psi^1\right) + \overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] \\
&\quad + \sin^2\phi^1\partial_t\overline{\psi}^2\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad \left.+\partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\right] \\
&\quad \left.-2i\sin^2\phi^1\left(\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1 + \dot{\phi}^2\psi^1\overline{\psi}^2\overline{\psi}^1\right)\right\} \\
&- 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 + \bar{\epsilon}\left\{\underline{-\sin\left(2\phi^1\right)\left(\dot{\phi}^2\right)^2\psi^1} + \partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\psi^2\right. \\
&\quad + i\left[2\sin\phi^1\cos\phi^1\overline{\psi}^2\psi^2\partial_t\psi^1 + 2\cos\left(2\phi^1\right)\psi^1\dot{\phi}^2\overline{\psi}^1\psi^2\right. \\
&\quad + \psi^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right] \\
&\quad + \sin\left(2\phi^1\right)\left[\dot{\phi}^2\left(\underline{-i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^1\psi^2\right)\psi^1 + i\dot{\phi}^1\dot{\phi}^2\psi^2 - \psi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] \\
&\quad + \sin^2\phi^1\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
&\quad \left.+\left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\right] \\
&\quad \left.-2i\sin^2\phi^1\left(\dot{\phi}^2\psi^1\psi^2\overline{\psi}^1 + \dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right)\right\}.
\end{aligned}
\tag{3.94}
$$

Here the underlined terms cancel each other. Also using the double angle Sine identity, we have

$$
\begin{aligned}
\frac{2\delta L}{r^2} = {} & 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + \epsilon\Big\{\underline{-\partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\overline{\psi}^2} \\
& + i\left[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + 2\cos\left(2\phi^1\right)\overline{\psi}^1\dot{\phi}^2\overline{\psi}^2\psi^1\right. \\
& \left. - \overline{\psi}^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right]\right. \\
& \left. + \sin\left(2\phi^1\right)\left[-i\dot{\phi}^1\dot{\phi}^2\overline{\psi}^2 - \cot\phi^1\dot{\phi}^2\overline{\psi}^1\overline{\psi}^2\psi^1 + \overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right]\right. \\
& \left. + \sin^2\phi^1\partial_t\overline{\psi}^2\left(\underline{i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\right. \\
& \left. + \partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\right] \\
& - 2i\sin^2\phi^1\left(\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1 + \dot{\phi}^2\psi^1\overline{\psi}^2\overline{\psi}^1\right)\Big\} \\
- 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 + {} & \bar{\epsilon}\Big\{\underline{\partial_t\left(2\sin^2\phi^1\dot{\phi}^2\right)\psi^2} \\
& + i\left[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 + 2\cos\left(2\phi^1\right)\psi^1\dot{\phi}^2\overline{\psi}^1\psi^2\right. \\
& \left. + \psi^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right]\right. \\
& \left. + \sin\left(2\phi^1\right)\left[-\cot\phi^1\dot{\phi}^2\overline{\psi}^1\psi^2\psi^1 + i\dot{\phi}^1\dot{\phi}^2\psi^2 - \psi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right]\right. \\
& \left. + \sin^2\phi^1\left(\underline{-i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2\right. \\
& \left. + \left(-i\dot{\phi}^2 - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\right] \\
& - 2i\sin^2\phi^1\left(\dot{\phi}^2\psi^1\psi^2\overline{\psi}^1 + \dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right)\Big\}.
\end{aligned}
\tag{3.95}
$$

Then using integration by parts on the $\pm i\dot{\phi}^2$ terms terms to isolate $\overline{\psi}^2$, we get

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + \dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 + \epsilon\Big\{\underline{-\partial_t\left(\sin^2\phi^1\dot{\phi}^2\right)\overline{\psi}^2} \\
&\quad + i\Big[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + 2\cos\left(2\phi^1\right)\overline{\psi}^1\dot{\phi}^2\overline{\psi}^2\psi^1 \\
&\quad - \overline{\psi}^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right] \\
&\quad + \sin\left(2\phi^1\right)\left[\underline{-i\dot{\phi}^1\dot{\phi}^2\overline{\psi}^2} - \cot\phi^1\dot{\phi}^2\overline{\psi}^1\overline{\psi}^2\psi^1 + \overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] \\
&\quad - \sin^2\phi^1\partial_t\overline{\psi}^2\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad + \partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(\underline{i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\Big] \\
&\quad - 2i\sin^2\phi^1\left(\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1 + \dot{\phi}^2\psi^1\overline{\psi}^2\overline{\psi}^1\right)\Big\} \\
&- 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 - \dot{\bar{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 + \bar{\epsilon}\Big\{\underline{\partial_t\left(\sin^2\phi^1\dot{\phi}^2\right)\psi^2} \\
&\quad + i\Big[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 + 2\cos\left(2\phi^1\right)\psi^1\dot{\phi}^2\overline{\psi}^1\psi^2 \\
&\quad + \psi^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right] \\
&\quad + \sin\left(2\phi^1\right)\left[-\cot\phi^1\dot{\phi}^2\overline{\psi}^1\psi^2\psi^1 + \underline{i\dot{\phi}^1\dot{\phi}^2\psi^2} - \psi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right] \\
&\quad - \sin^2\phi^1\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
&\quad + \left(\underline{-i\dot{\phi}^2} - \cot\phi^1\overline{\psi}^1\psi^2 - \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\Big] \\
&\quad - 2i\sin^2\phi^1\left(\dot{\phi}^2\psi^1\psi^2\overline{\psi}^1 + \dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right)\Big\}.
\end{aligned}
\tag{3.96}
$$

Then using integration by parts again on the $\pm i\dot{\phi}^2$, we get

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 + \epsilon\Big\{i\Big[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 \\
&\quad + \underline{2\cos\left(2\phi^1\right)\overline{\psi}^1\dot{\phi}^2\overline{\psi}^2\psi^1} - \overline{\psi}^2\partial_t\Big[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\Big] \\
&\quad + \sin\left(2\phi^1\right)\Big[-\underline{\cot\phi^1\dot{\phi}^2\overline{\psi}^1\overline{\psi}^2\psi^1} + \overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\Big] \\
&\quad - \sin^2\phi^1\partial_t\overline{\psi}^2\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad - \partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\Big] \\
&\quad - 2i\sin^2\phi^1\left(\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1 + \underline{\dot{\phi}^2\psi^1\overline{\psi}^2\overline{\psi}^1}\right)\Big\} \\
&- 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\bar{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 + \bar{\epsilon}\Big\{i\Big[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 \\
&\quad + \underline{2\cos\left(2\phi^1\right)\psi^1\dot{\phi}^2\overline{\psi}^1\psi^2} + \psi^2\partial_t\Big[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\Big] \\
&\quad + \sin\left(2\phi^1\right)\Big[-\underline{\cot\phi^1\dot{\phi}^2\overline{\psi}^1\psi^2\psi^1} - \psi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\Big] \\
&\quad - \sin^2\phi^1\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
&\quad - \left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\Big] \\
&\quad - 2i\sin^2\phi^1\left(\underline{\dot{\phi}^2\psi^1\psi^2\overline{\psi}^1} + \dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right)\Big\}.
\end{aligned}
\tag{3.97}
$$

Here the underline terms cancel each other out as they did earlier.

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 \\
&+ \epsilon\left\{ i\left[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 - \underline{\overline{\psi}^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right]}\right.\right. \\
&\quad + \sin\left(2\phi^1\right)\overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right) - \sin^2\phi^1\partial_t\overline{\psi}^2\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad \left.\left. -\partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\right] - 2i\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\right\} \\
&- 2\dot{\overline{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\overline{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 \qquad (3.98) \\
&+ \overline{\epsilon}\left\{ i\left[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 + \underline{\psi^2\partial_t\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right]}\right.\right. \\
&\quad - \sin\left(2\phi^1\right)\psi^1\left(\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right) - \sin^2\phi^1\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
&\quad \left.\left. -\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\right] - 2i\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right\}
\end{aligned}
$$

Now using integration by parts on the underlined terms gives us

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 - i\dot{\epsilon}\sin\left(2\phi^1\right)\overline{\psi}^2\overline{\psi}^1\psi^2 \\
&+ \epsilon\left\{ i\left[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \partial_t\overline{\psi}^2\left[\sin\left(2\phi^1\right)\left(\overline{\psi}^2\psi^1 - \underline{\overline{\psi}^1\psi^2}\right)\right]\right.\right. \\
&\quad + \sin\left(2\phi^1\right)\overline{\psi}^1\left(\overline{\psi}^2\partial_t\psi^2 - \underline{\partial_t\overline{\psi}^2\psi^2}\right) - \sin^2\phi^1\partial_t\overline{\psi}^2\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad \left.\left. -\partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\right] - 2i\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\right\} \\
&- 2\dot{\overline{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\overline{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 - i\dot{\overline{\epsilon}}\sin\left(2\phi^1\right)\psi^2\overline{\psi}^2\psi^1 \qquad (3.99) \\
&+ \overline{\epsilon}\left\{ i\left[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 - \partial_t\psi^2\left[\sin\left(2\phi^1\right)\left(\underline{\overline{\psi}^2\psi^1} - \overline{\psi}^1\psi^2\right)\right]\right.\right. \\
&\quad - \sin\left(2\phi^1\right)\psi^1\left(\underline{\overline{\psi}^2\partial_t\psi^2} - \partial_t\overline{\psi}^2\psi^2\right) - \sin^2\phi^1\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
&\quad \left.\left. -\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\left(\sin^2\phi^1\psi^2\right)\right] - 2i\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right\}.
\end{aligned}
$$

Now using integration by parts on the underlined terms gives us

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\epsilon\dot{\phi}^1\overline{\psi}^1 + 2\epsilon\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 - i\epsilon\sin\left(2\phi^1\right)\overline{\psi}^2\overline{\psi}^1\psi^2 \\
&\quad + \epsilon\left\{i\left[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \partial_t\overline{\psi}^2\sin\left(2\phi^1\right)\overline{\psi}^2\psi^1 + \sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\partial_t\psi^2\right.\right. \\
&\quad\quad - \sin^2\phi^1\partial_t\overline{\psi}^2\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right) \\
&\quad\quad \left.\left.\underline{-\partial_t\left(\sin^2\phi^1\overline{\psi}^2\right)}\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\right] - 2i\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\right\} \\
&\quad - 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\bar{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 - i\dot{\bar{\epsilon}}\sin\left(2\phi^1\right)\psi^2\overline{\psi}^2\psi^1 \qquad (3.100) \\
&\quad + \bar{\epsilon}\left\{i\left[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 + \partial_t\psi^2\sin\left(2\phi^1\right)\overline{\psi}^1\psi^2 + \sin\left(2\phi^1\right)\psi^1\partial_t\overline{\psi}^2\psi^2\right.\right. \\
&\quad\quad - \sin^2\phi^1\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 \\
&\quad\quad \left.\left. - \left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\underline{\partial_t\left(\sin^2\phi^1\psi^2\right)}\right] - 2i\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right\}.
\end{aligned}
$$

Expanding the derivatives here allows us to say

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\epsilon\dot{\phi}^1\overline{\psi}^1 + 2\epsilon\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 - i\epsilon\sin\left(2\phi^1\right)\overline{\psi}^2\overline{\psi}^1\psi^2 \\
&\quad + \epsilon\left\{i\left[\sin\left(2\phi^1\right)\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \partial_t\overline{\psi}^2\sin\left(2\phi^1\right)\overline{\psi}^2\psi^1 + \sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\partial_t\psi^2\right.\right. \\
&\quad\quad -2\sin^2\phi^1\partial_t\overline{\psi}^2\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right) - \sin\left(2\phi^1\right)\cot\phi^1\dot{\phi}^1\overline{\psi}^2\overline{\psi}^1\psi^2\Big] \\
&\quad\quad \left.-2i\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\right\} \\
&\quad - 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\bar{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 - i\dot{\bar{\epsilon}}\sin\left(2\phi^1\right)\psi^2\overline{\psi}^2\psi^1 \qquad (3.101) \\
&\quad + \bar{\epsilon}\left\{i\left[\sin\left(2\phi^1\right)\overline{\psi}^2\psi^2\partial_t\psi^1 + \partial_t\psi^2\sin\left(2\phi^1\right)\overline{\psi}^1\psi^2 + \sin\left(2\phi^1\right)\psi^1\partial_t\overline{\psi}^2\psi^2\right.\right. \\
&\quad\quad -2\sin^2\phi^1\left(\cot\phi^1\overline{\psi}^1\psi^2 + \cot\phi^1\overline{\psi}^2\psi^1\right)\partial_t\psi^2 - \sin\left(2\phi^1\right)\cot\phi^1\dot{\phi}^1\overline{\psi}^2\psi^1\psi^2\Big] \\
&\quad\quad \left.-2i\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right\}.
\end{aligned}
$$

Using the double angle sine identity, we get

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 - i\dot{\epsilon}\sin\left(2\phi^1\right)\overline{\psi}^2\overline{\psi}^1\psi^2 \\
&\quad + \epsilon\left\{i\sin\left(2\phi^1\right)\left[\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \partial_t\overline{\psi}^2\overline{\psi}^2\psi^1 + \overline{\psi}^1\overline{\psi}^2\partial_t\psi^2\right.\right. \\
&\quad \left.\left. -\partial_t\overline{\psi}^2\overline{\psi}^1\psi^2 - \partial_t\overline{\psi}^2\overline{\psi}^2\psi^1 - \cot\phi^1\dot{\phi}^1\overline{\psi}^2\overline{\psi}^1\psi^2\right] - 2i\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\right\} \\
&\quad - 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\bar{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 - i\dot{\bar{\epsilon}}\sin\left(2\phi^1\right)\psi^2\overline{\psi}^2\psi^1 \\
&\quad + \bar{\epsilon}\left\{i\sin\left(2\phi^1\right)\left[\overline{\psi}^2\psi^2\partial_t\psi^1 + \partial_t\psi^2\overline{\psi}^1\psi^2 + \psi^1\partial_t\overline{\psi}^2\psi^2\right.\right. \\
&\quad \left.\left. -\overline{\psi}^1\psi^2\partial_t\psi^2 - \overline{\psi}^2\psi^1\partial_t\psi^2 - \cot\phi^1\dot{\phi}^1\overline{\psi}^2\psi^1\psi^2\right] - 2i\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right\}.
\end{aligned}
\tag{3.102}
$$

This allows us to simplify the equation, to get

$$
\begin{aligned}
\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 - i\dot{\epsilon}\sin\left(2\phi^1\right)\overline{\psi}^2\overline{\psi}^1\psi^2 \\
&\quad + \epsilon\left\{i\sin\left(2\phi^1\right)\left[\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\overline{\psi}^1\psi^2 - \cot\phi^1\dot{\phi}^1\overline{\psi}^2\overline{\psi}^1\psi^2\right]\right. \\
&\quad \left. -2i\sin^2\phi^1\dot{\phi}^1\overline{\psi}^2\psi^2\overline{\psi}^1\right\} \\
&\quad - 2\dot{\bar{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\bar{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 - i\dot{\bar{\epsilon}}\sin\left(2\phi^1\right)\psi^2\overline{\psi}^2\psi^1 \\
&\quad \bar{\epsilon}\left\{i\sin\left(2\phi^1\right)\left[\overline{\psi}^2\psi^2\partial_t\psi^1 + \psi^1\partial_t\overline{\psi}^2\psi^2 - \overline{\psi}^2\psi^1\partial_t\psi^2 - \cot\phi^1\dot{\phi}^1\overline{\psi}^2\psi^1\psi^2\right]\right. \\
&\quad \left. -2i\sin^2\phi^1\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right\}.
\end{aligned}
\tag{3.103}
$$

Here we can rearrange this to get

$$\begin{aligned}\frac{2\delta L}{r^2} &= 2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2 + i\dot{\epsilon}\sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\psi^2 \\ &+ i\epsilon\left[\sin\left(2\phi^1\right)\left(\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 + \overline{\psi}^1\partial_t\overline{\psi}^2\psi^2\right)\right. \\ &\left.+2\left(\cos^2\phi^1 - \sin^2\phi^1\right)\dot{\phi}^1\overline{\psi}^1\overline{\psi}^2\psi^2\right] \\ &- 2\dot{\overline{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\overline{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2 + i\dot{\overline{\epsilon}}\sin\left(2\phi^1\right)\psi^1\overline{\psi}^2\psi^2 \\ &+ i\overline{\epsilon}\left[\sin\left(2\phi^1\right)\left(\partial_t\psi^1\overline{\psi}^2\psi^2 + \psi^1\partial_t\overline{\psi}^2\psi^2 + \psi^1\overline{\psi}^2\partial_t\psi^2\right)\right. \\ &\left.+2\left(\cos^2\phi^1 - \sin^2\phi^1\right)\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right].\end{aligned} \tag{3.104}$$

Using the $\cos\left(2\phi^1\right) = \cos^2\phi^1 - \sin^2\phi^1$, we have

$$\begin{aligned}\frac{2\delta L}{r^2} &= \underline{2\dot{\epsilon}\dot{\phi}^1\overline{\psi}^1 + 2\dot{\epsilon}\sin^2\phi^1\dot{\phi}^2\overline{\psi}^2} + i\dot{\epsilon}\sin\left(2\phi^1\right)\overline{\psi}^1\overline{\psi}^2\psi^2 \\ &+ i\epsilon\left[\sin\left(2\phi^1\right)\left(\partial_t\overline{\psi}^1\overline{\psi}^2\psi^2 + \overline{\psi}^1\overline{\psi}^2\partial_t\psi^2 + \overline{\psi}^1\partial_t\overline{\psi}^2\psi^2\right) + 2\cos\left(2\phi^1\right)\dot{\phi}^1\overline{\psi}^1\overline{\psi}^2\psi^2\right] \\ &\underline{-2\dot{\overline{\epsilon}}\dot{\phi}^1\psi^1 - 2\dot{\overline{\epsilon}}\sin^2\phi^1\dot{\phi}^2\psi^2} + i\dot{\overline{\epsilon}}\sin\left(2\phi^1\right)\psi^1\overline{\psi}^2\psi^2 \\ &+ i\overline{\epsilon}\left[\sin\left(2\,\phi^1\right)\left(\partial_t\psi^1\overline{\psi}^2\psi^2 + \psi^1\partial_t\overline{\psi}^2\psi^2 + \psi^1\overline{\psi}^2\partial_t\psi^2\right) + 2\cos\left(2\phi^1\right)\dot{\phi}^1\psi^1\overline{\psi}^2\psi^2\right].\end{aligned} \tag{3.105}$$

Noting that the everything other than the two underlined terms are part of a total time derivative, we have that

$$\delta L = \dot{\epsilon}r^2\left(\dot{\phi}^1\overline{\psi}^1 + \sin^2\phi^1\dot{\phi}^2\overline{\psi}^2\right) - \dot{\overline{\epsilon}}r^2\left(\dot{\phi}^1\psi^1 + \sin^2\phi^1\dot{\phi}^2\psi^2\right). \tag{3.106}$$

Now integrating everything over time and pulling out an $-i$,

$$\delta L dt = -i\int\left\{\dot{\epsilon}ir^2\left(\dot{\phi}^1\overline{\psi}^1 + \sin^2\phi^1\dot{\phi}^2\overline{\psi}^2\right) - \dot{\overline{\epsilon}}ir^2\left(\dot{\phi}^1\psi^1 + \sin^2\phi^1\dot{\phi}^2\psi^2\right)\right\}dt, \tag{3.107}$$

we have that

$$Q = ir^2\left(\dot{\phi}^1\overline{\psi}^1 + \sin^2\phi^1\dot{\phi}^2\overline{\psi}^2\right) \tag{3.108}$$

$$Q^\dagger = -ir^2\left(\dot{\phi}^1\psi^1 + \sin^2\phi^1\dot{\phi}^2\psi^2\right) \tag{3.109}$$

which is exactly what we calculated before.

Now to check the Noether charge F, we can look at Lagrangian from eq.(3.68) under time-dependant phase rotation of the fermions.

$$\begin{aligned}\frac{2(L+\delta L)}{r^2} = &\left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 + i\left[e^{i\gamma}\overline{\psi}^1\partial_t\left(e^{-i\gamma}\psi^1\right) - \partial_t\left(e^{i\gamma}\overline{\psi}^1\right)e^{-i\gamma}\psi^1\right.\\ &+\sin\left(2\phi^1\right)\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right) + \sin^2\phi^1\left(e^{i\gamma}\overline{\psi}^2\partial_t\left(e^{-i\gamma}\psi^2\right)\right.\\ &\left.\left.-\partial_t\left(e^{i\gamma}\overline{\psi}^2\right)e^{-i\gamma}\psi^2\right)\right] - 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.110}$$

Evaluating the derivatives gives us

$$\begin{aligned}\frac{2(L+\delta L)}{r^2} = &\left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2\\ &+ i\left[-i\dot{\gamma}\overline{\psi}^1\psi^1 + \overline{\psi}^1\partial_t\psi^1 - i\dot{\gamma}\overline{\psi}^1\psi^1 - \partial_t\overline{\psi}^1\psi^1 + \sin\left(2\phi^1\right)\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right)\right.\\ &\left.+\sin^2\phi^1\left(-i\dot{\gamma}\overline{\psi}^2\psi^2 + \overline{\psi}^2\partial_t\psi^2 - i\dot{\gamma}\overline{\psi}^2\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\right]\\ &- 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.111}$$

Now combining like terms gives us

$$\begin{aligned}\frac{2(L+\delta L)}{r^2} = &\left(\dot{\phi}^1\right)^2 + \sin^2\phi^1\left(\dot{\phi}^2\right)^2 + i\left[\underline{-2i\dot{\gamma}\overline{\psi}^1\psi^1} + \overline{\psi}^1\partial_t\psi^1 - \partial_t\overline{\psi}^1\psi^1\right.\\ &+\sin\left(2\phi^1\right)\dot{\phi}^2\left(\overline{\psi}^2\psi^1 - \overline{\psi}^1\psi^2\right) + \sin^2\phi^1\left(\underline{-2i\dot{\gamma}\overline{\psi}^2\psi^2} + \overline{\psi}^2\partial_t\psi^2 - \partial_t\overline{\psi}^2\psi^2\right)\Big]\\ &- 2\sin^2\left(\phi^1\right)\psi^1\overline{\psi}^2\psi^2\overline{\psi}^1.\end{aligned} \tag{3.112}$$

Since this only adds two terms to the Lagrangian we have that

$$\delta L = \dot{\gamma}\, r^2 \left(\overline{\psi}^1 \psi^1 + \sin^2 \phi^1 \overline{\psi}^2 \psi^2 \right). \tag{3.113}$$

So take an integral over time

$$\delta L dt = \int \dot{\gamma}\, r^2 \left(\overline{\psi}^1 \psi^1 + \sin^2 \phi^1 \overline{\psi}^2 \psi^2 \right) dt \tag{3.114}$$

means that

$$F = r^2 \left(\overline{\psi}^1 \psi^1 + \sin^2 \phi^1 \overline{\psi}^2 \psi^2 \right), \tag{3.115}$$

which matches the earlier calculation.

3.3 Superfield Formalism

Under the Superfield formalism in 1D, the SUSY Lagrangian can be derived using

$$L_{SUSY} = \frac{1}{2} g_{IJ}(\Phi) \overline{D} \Phi^I D \Phi^J, \tag{3.116}$$

where Φ is a superfield, D and $\overline{D}$ are operators defined as

$$D = \partial_\theta - i\overline{\theta}\partial_t \tag{3.117}$$

$$\overline{D} = -\partial_{\overline{\theta}} + i\theta\partial_t \tag{3.118}$$

where

$$\partial_\theta \theta \overline{\theta} = -\overline{\theta}. \tag{3.119}$$

Here θ and $\overline{\theta}$ are fermionic variables that augment physical space and are orthogonal to the 1D space. For our purposes, we take the i^{th} component of the superfield Φ to

be

$$\Phi^I = \phi^I + \theta\psi^I + \overline{\theta}\,\overline{\psi}^I + i\theta\overline{\theta}F^I, \tag{3.120}$$

where F^I is a bosonic function representing an auxiliary field to be determined later. The initial ordering of the θ and $\overline{\theta}$ does not matter as long as we stick to the ordering through out the derivation.

Now using the operators on a component of the superfield, we get

$$\overline{D}\Phi^I = \overline{\psi}^I - i\theta F^I + i\theta\dot{\phi}^I + i\theta\overline{\theta}\partial_t\overline{\psi}^I \tag{3.121}$$

$$D\Phi^J = -\psi^J - i\overline{\theta}F^J - i\overline{\theta}\dot{\phi}^J + i\theta\overline{\theta}\partial_t\psi^J. \tag{3.122}$$

Then multiplying them together yields

$$\begin{aligned}\overline{D}\Phi^I D\Phi^J =& -\overline{\psi}^I\psi^J - i\overline{\theta\psi}^I F^J + i\overline{\theta\psi}^I\dot{\phi}^J + i\theta\overline{\theta\psi}^I\partial_t\psi^J \\ &- i\theta F^I\psi^J - \theta\overline{\theta}F^I F^J - \theta\overline{\theta}F^I\dot{\phi}^J - i\theta\dot{\phi}^I\psi^J \\ &+ \theta\overline{\theta}\dot{\phi}^I F^J + \theta\overline{\theta}\dot{\phi}^I\dot{\phi}^J - i\theta\overline{\theta}\partial_t\overline{\psi}^I\psi^J.\end{aligned} \tag{3.123}$$

Now taking $g_{IJ}(\Phi)$ as an expansion to be

$$g_{IJ}(\Phi) = g_{IJ} + \theta\partial_\gamma g_{IJ}\psi^\gamma + \overline{\theta}\partial_\gamma g_{IJ}\overline{\psi}^\gamma + \theta\overline{\theta}\left(i\partial_\gamma g_{IJ}F^\gamma - \partial_\gamma\partial_\delta g_{IJ}\psi^\gamma\overline{\psi}^\delta\right), \tag{3.124}$$

and keeping only the $\theta\overline{\theta}$ terms, the Lagrangian becomes

$$\begin{aligned}2L =& -i\partial_\gamma g_{IJ}F^\gamma\overline{\psi}^I\psi^J + \partial_\gamma\partial_\delta g_{IJ}\psi^\gamma\overline{\psi}^\delta\overline{\psi}^I\psi^J - i\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I F^J \\ &+ i\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J + ig_{IJ}\overline{\psi}^I\partial_t\psi^J + i\partial_\gamma g_{IJ}\overline{\psi}^\gamma F^I\psi^J - g_{IJ}F^I F^J \\ &- \underline{g_{IJ}F^I\dot{\phi}^J} + i\partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J + \underline{g_{IJ}\dot{\phi}^I F^J} + g_{IJ}\dot{\phi}^I\dot{\phi}^J - ig_{IJ}\partial_t\overline{\psi}^I\psi^J.\end{aligned} \tag{3.125}$$

Assuming that g_{IJ} is symmetric, the two underlined terms cancel each other.

$$\begin{aligned} 2L = &- i\partial_\gamma g_{IJ} F^\gamma \overline{\psi}^I \psi^J + \partial_\gamma \partial_\delta g_{IJ} \psi^\gamma \overline{\psi}^\delta \overline{\psi}^I \psi^J - i\partial_\gamma g_{IJ} \psi^\gamma \overline{\psi}^I F^J \\ &+ i\partial_\gamma g_{IJ} \psi^\gamma \overline{\psi}^I \dot{\phi}^J + i g_{IJ} \overline{\psi}^I \partial_t \psi^J + i\partial_\gamma g_{IJ} \overline{\psi}^\gamma F^I \psi^J - g_{IJ} F^I F^J \\ &+ i\partial_\gamma g_{IJ} \overline{\psi}^\gamma \dot{\phi}^I \psi^J + g_{IJ} \dot{\phi}^I \dot{\phi}^J - i g_{IJ} \partial_t \overline{\psi}^I \psi^J. \end{aligned} \tag{3.126}$$

Now to find F^α, we take the variation of Lagrangian with respect to the bosonic functions F^α and set it equal to zero.

$$- i\partial_\gamma g_{IJ} \overline{\psi}^I \psi^J \delta F^\gamma - i\partial_\gamma g_{IJ} \psi^\gamma \overline{\psi}^I \delta F^J + i\partial_\gamma g_{IJ} \overline{\psi}^\gamma \psi^J \delta F^I - 2 g_{IJ} F^I \delta F^J = 0 \tag{3.127}$$

Here rearranging the indices on the first three terms gives us

$$2 g_{IJ} F^I \delta F^J = -i\partial_J g_{I\gamma} \overline{\psi}^I \psi^\gamma \delta F^J - i\partial_I g_{\gamma J} \psi^I \overline{\psi}^\gamma \delta F^J + i\partial_\gamma g_{IJ} \overline{\psi}^\gamma \psi^I \delta F^J, \tag{3.128}$$

or more simply as

$$2 g_{IJ} F^I = -i\partial_J g_{I\gamma} \overline{\psi}^I \psi^\gamma - i\partial_I g_{\gamma J} \psi^I \overline{\psi}^\gamma + i\partial_\gamma g_{IJ} \overline{\psi}^\gamma \psi^I. \tag{3.129}$$

Now swapping the order of the fermionic variables in the second term yields a negative sign

$$2 g_{IJ} F^I = -i\partial_J g_{I\gamma} \overline{\psi}^I \psi^\gamma + i\partial_I g_{\gamma J} \overline{\psi}^\gamma \psi^I + i\partial_\gamma g_{IJ} \overline{\psi}^\gamma \psi^I. \tag{3.130}$$

Then noting again that the metric tensor is symmetric in the first te, we can say that

$$g_{IJ} F^I = i \left[\frac{1}{2} \left(-\partial_J g_{I\gamma} + \partial_I g_{\gamma J} + \partial_\gamma g_{IJ} \right) \right] \overline{\psi}^\gamma \psi^I. \tag{3.131}$$

Here we can note that inside the bracket is a Christoffel symbol of the first kind.

$$g_{IJ} F^I = i \Gamma_{\gamma IJ} \overline{\psi}^\gamma \psi^I. \tag{3.132}$$

Finally we can multiply both sides of the equation with $g^{\alpha J}$ to get a Christoffel symbol

of the second kind.

$$F^{\alpha} = i\Gamma^{\alpha}_{\gamma I}\overline{\psi}^{\gamma}\psi^{I}. \tag{3.133}$$

After finding F^{α}, we can use this inside the Lagrangian to get

$$\begin{aligned} 2L =& \underline{\partial_{\gamma} g_{IJ}\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J}} + \partial_{\gamma}\partial_{\delta} g_{IJ}\psi^{\gamma}\overline{\psi}^{\delta}\overline{\psi}^{I}\psi^{J} + \underline{\partial_{\gamma} g_{IJ}\Gamma^{J}_{\alpha\beta}\psi^{\gamma}\overline{\psi}^{I}\overline{\psi}^{\alpha}\psi^{\beta}} \\ &+ i\partial_{\gamma} g_{IJ}\psi^{\gamma}\overline{\psi}^{I}\dot{\phi}^{J} + ig_{IJ}\overline{\psi}^{I}\partial_{t}\psi^{J} - \underline{\partial_{\gamma} g_{IJ}\Gamma^{I}_{\alpha\beta}\overline{\psi}^{\gamma}\overline{\psi}^{\alpha}\psi^{\beta}\psi^{J}} \\ &+ g_{IJ}\Gamma^{I}_{\alpha\beta}\Gamma^{J}_{\gamma\delta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{\gamma}\psi^{\delta} + i\partial_{\gamma} g_{IJ}\overline{\psi}^{\gamma}\dot{\phi}^{I}\psi^{J} + g_{IJ}\dot{\phi}^{I}\dot{\phi}^{J} - ig_{IJ}\partial_{t}\overline{\psi}^{I}\psi^{J}. \end{aligned} \tag{3.134}$$

Now looking more closely at the underlined terms, we can swap the order of the fermionic variables in the last two terms to have the same form as the first to get

$$\begin{aligned} &\partial_{\gamma} g_{IJ}\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J} + \partial_{\gamma} g_{IJ}\Gamma^{J}_{\alpha\beta}\psi^{\gamma}\overline{\psi}^{I}\overline{\psi}^{\alpha}\psi^{\beta} - \partial_{\gamma} g_{IJ}\Gamma^{I}_{\alpha\beta}\overline{\psi}^{\gamma}\overline{\psi}^{\alpha}\psi^{\beta}\psi^{J} \\ &\quad = \partial_{\gamma} g_{IJ}\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J} - \partial_{\gamma} g_{IJ}\Gamma^{J}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{\gamma} - \partial_{\gamma} g_{IJ}\Gamma^{I}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{\gamma}\psi^{J}. \end{aligned} \tag{3.135}$$

Similarly, we can relabel the indices in the last two terms to have the same fo as the first to get

$$\begin{aligned} &\partial_{\gamma} g_{IJ}\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J} + \partial_{\gamma} g_{IJ}\Gamma^{J}_{\alpha\beta}\psi^{\gamma}\overline{\psi}^{I}\overline{\psi}^{\alpha}\psi^{\beta} - \partial_{\gamma} g_{IJ}\Gamma^{I}_{\alpha\beta}\overline{\psi}^{\gamma}\overline{\psi}^{\alpha}\psi^{\beta}\psi^{J} \\ &\quad = -\left(-\partial_{\gamma} g_{IJ} + \partial_{J} g_{I\gamma} + \partial_{I} g_{\gamma J}\right)\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J}. \end{aligned} \tag{3.136}$$

From here we can see in the parenthesis that there are two Christoffel symbols of the first kind.

$$\begin{aligned} &\partial_{\gamma} g_{IJ}\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J} + \partial_{\gamma} g_{IJ}\Gamma^{J}_{\alpha\beta}\psi^{\gamma}\overline{\psi}^{I}\overline{\psi}^{\alpha}\psi^{\beta} - \partial_{\gamma} g_{IJ}\Gamma^{I}_{\alpha\beta}\overline{\psi}^{\gamma}\overline{\psi}^{\alpha}\psi^{\beta}\psi^{J} \\ &\quad = -2\Gamma_{IJ\gamma}\Gamma^{\gamma}_{\alpha\beta}\overline{\psi}^{\alpha}\psi^{\beta}\overline{\psi}^{I}\psi^{J}. \end{aligned} \tag{3.137}$$

Now using $\Gamma_{IJ\gamma} = g_{\gamma m}\Gamma^m_{IJ}$, we can say that

$$\begin{aligned}\partial_\gamma g_{IJ}\Gamma^\gamma_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^I\psi^J + \partial_\gamma g_{IJ}\Gamma^J_{\alpha\beta}\psi^\gamma\overline{\psi}^I\overline{\psi}^\alpha\psi^\beta - \partial_\gamma g_{IJ}\Gamma^I_{\alpha\beta}\overline{\psi}^\gamma\overline{\psi}^\alpha\psi^\beta\psi^J \\ = -2g_{\gamma m}\Gamma^m_{IJ}\Gamma^\gamma_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^I\psi^J.\end{aligned} \tag{3.138}$$

Using this, the Lagrangian becomes

$$\begin{aligned}2L =& -2g_{IJ}\Gamma^I_{\gamma\delta}\Gamma^J_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^\gamma\psi^\delta + \partial_\gamma\partial_\delta g_{IJ}\psi^\gamma\overline{\psi}^\delta\overline{\psi}^I\psi^J + i\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J + ig_{IJ}\overline{\psi}^I\partial_t\psi^J \\ &+ g_{IJ}\Gamma^I_{\alpha\beta}\Gamma^J_{\gamma\delta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^\gamma\psi^\delta + i\partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J + g_{IJ}\dot{\phi}^I\dot{\phi}^J - ig_{IJ}\partial_t\overline{\psi}^I\psi^J.\end{aligned} \tag{3.139}$$

For simplicity, we can rewrite this as

$$\begin{aligned}2L =& g_{IJ}\dot{\phi}^I\dot{\phi}^J + i\left(\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J + \underline{g_{IJ}\overline{\psi}^I\partial_t\psi^J} + \partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J - \underline{g_{IJ}\partial_t\overline{\psi}^I\psi^J}\right) \\ &+ \partial_\gamma\partial_\delta g_{IJ}\psi^\gamma\overline{\psi}^\delta\overline{\psi}^I\psi^J - g_{IJ}\Gamma^I_{\alpha\beta}\Gamma^J_{\gamma\delta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^\gamma\psi^\delta.\end{aligned} \tag{3.140}$$

Now using integration by parts on the imaginary portion of the Lagrangian for the underlined terms we get

$$-g_{IJ}\partial_t\overline{\psi}^I\psi^J = \partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J + g_{IJ}\overline{\psi}^I\partial_t\psi^J \tag{3.141}$$

$$g_{IJ}\overline{\psi}^I\partial_t\psi^J = -\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J. \tag{3.142}$$

Now for each of these equations, we can break up the first term into two pieces and rearrange the indices to get

$$-g_{IJ}\partial_t\overline{\psi}^I\psi^J = \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J - \frac{1}{2}\partial_I g_{\gamma J}\dot{\phi}^I\psi^J\overline{\psi}^\gamma + g_{IJ}\overline{\psi}^I\partial_t\psi^J \tag{3.143}$$

$$g_{IJ}\overline{\psi}^I\partial_t\psi^J = -\frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J. \tag{3.144}$$

Then adding these two equations together gives us

$$g_{IJ}\overline{\psi}^I\partial_t\psi^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J = -\frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J + g_{IJ}\overline{\psi}^I\partial_t\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\psi^I\overline{\psi}^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J \tag{3.145}$$

Similarly, we can break up the non-underlined terms of the imaginary portion of the Lagrangian into two pieces and rearrange the indices to get

$$\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J = -\frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\overline{\psi}^J\psi^\gamma + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma \tag{3.146}$$

$$\partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J = -\frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\psi^J\overline{\psi}^\gamma + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma. \tag{3.147}$$

Now adding up these last three equations gives us

$$\begin{aligned}
&\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I\partial_t\psi^J + \partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J \\
&\quad = \underline{-\frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J} + g_{IJ}\overline{\psi}^I\partial_t\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\psi^I\overline{\psi}^J - \underline{g_{IJ}\partial_t\overline{\psi}^I\psi^J} \\
&\quad\quad - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\overline{\psi}^J\psi^\gamma + \underline{\frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma} - \underline{\frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\psi^J\overline{\psi}^\gamma} + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma.
\end{aligned} \tag{3.148}$$

Then grouping the underlined terms and the non-underlined terms together, we can say that

$$\begin{aligned}
&-g_{IJ}\partial_t\overline{\psi}^I\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\psi^J\overline{\psi}^\gamma \\
&\quad = -g_{IJ}\partial_t\overline{\psi}^I\psi^J + \left[\frac{1}{2}\left(\partial_I g_{\gamma J} + \partial_J g_{I\gamma} - \partial_\gamma g_{IJ}\right)\right]\dot{\phi}^I\psi^J\overline{\psi}^\gamma
\end{aligned} \tag{3.149}$$

$$\begin{aligned}
&g_{IJ}\overline{\psi}^I\partial_t\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\psi^I\overline{\psi}^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\overline{\psi}^J\psi^\gamma + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma \\
&\quad = g_{IJ}\overline{\psi}^I\partial_t\psi^J - \left[\frac{1}{2}\left(\partial_I g_{J\gamma} - \partial_\gamma g_{IJ} + \partial_J g_{I\gamma}\right)\right]\dot{\phi}^I\overline{\psi}^J\psi^\gamma.
\end{aligned} \tag{3.150}$$

Now inside the brackets, we have Christoffel symbols of the first kind.

$$-g_{IJ}\partial_t\overline{\psi}^I\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\psi^J\overline{\psi}^\gamma \tag{3.151}$$
$$= -g_{IJ}\partial_t\overline{\psi}^I\psi^J + \Gamma_{IJ\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma$$

$$g_{IJ}\overline{\psi}^I\partial_t\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\psi^I\overline{\psi}^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\overline{\psi}^J\psi^\gamma + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma \tag{3.152}$$
$$= g_{IJ}\overline{\psi}^I\partial_t\psi^J + \Gamma_{IJ\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma.$$

Now using $\Gamma_{IJ\gamma} = g_{\gamma m}\Gamma^m_{IJ}$, we can say that

$$-g_{IJ}\partial_t\overline{\psi}^I\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\psi^J\overline{\psi}^\gamma \tag{3.153}$$
$$= -g_{IJ}\left(\partial_t\overline{\psi}^I + \Gamma^I_{\alpha\beta}\dot{\phi}^\alpha\overline{\psi}^\beta\right)\psi^J$$

$$g_{IJ}\overline{\psi}^I\partial_t\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\psi^I\overline{\psi}^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\overline{\psi}^J\psi^\gamma + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma \tag{3.154}$$
$$= g_{IJ}\overline{\psi}^I\left(\partial_t\psi^J + \Gamma^J_{\alpha\beta}\dot{\phi}^\alpha\psi^\beta\right).$$

By noting the the terms inside the parenthesis are covariant time derivatives D_t, we get

$$-g_{IJ}\partial_t\overline{\psi}^I\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\overline{\psi}^I\psi^J + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\psi^J\overline{\psi}^\gamma - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\psi^J\overline{\psi}^\gamma = -g_{IJ}D_t\overline{\psi}^I\psi^J \tag{3.155}$$

$$g_{IJ}\overline{\psi}^I\partial_t\psi^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^\gamma\psi^I\overline{\psi}^J - \frac{1}{2}\partial_\gamma g_{IJ}\dot{\phi}^I\overline{\psi}^J\psi^\gamma + \frac{1}{2}\partial_J g_{I\gamma}\dot{\phi}^I\overline{\psi}^J\psi^\gamma = g_{IJ}\overline{\psi}^I D_t\psi^J. \tag{3.156}$$

This makes the imaginary portion of the Lagrangian becomes

$$\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I\partial_t\psi^J + \partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J \tag{3.157}$$
$$= -g_{IJ}D_t\overline{\psi}^I\psi^J + g_{IJ}\overline{\psi}^I D_t\psi^J.$$

By integration by parts, we have

$$\begin{aligned}\partial_\gamma g_{IJ}\psi^\gamma\overline{\psi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I\partial_t\psi^J + \partial_\gamma g_{IJ}\overline{\psi}^\gamma\dot{\phi}^I\psi^J - g_{IJ}\partial_t\overline{\psi}^I\psi^J \\ = g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J.\end{aligned} \tag{3.158}$$

Therefore the Lagrangian becomes

$$\begin{aligned}2L =& g_{IJ}\dot{\phi}^I\dot{\phi}^J + i\left(g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J\right) \\ &+ \partial_\gamma\partial_\delta g_{IJ}\psi^\gamma\overline{\psi}^\delta\overline{\psi}^I\psi^J - g_{IJ}\Gamma^I_{\alpha\beta}\Gamma^J_{\gamma\delta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^\gamma\psi^\delta.\end{aligned} \tag{3.159}$$

Now for the terms with second derivative of the metric tensor, we can it up into two terms and rearrange the indices to get

$$\begin{aligned}2L =& g_{IJ}\dot{\phi}^I\dot{\phi}^J + i\left(g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J\right) \\ &+ \frac{1}{2}\left(\partial_\gamma\partial_\delta g_{IJ} + \partial_I\partial_J g_{\gamma\delta}\right)\psi^\gamma\overline{\psi}^\delta\overline{\psi}^I\psi^J - g_{IJ}\Gamma^I_{\alpha\beta}\Gamma^J_{\gamma\delta}\overline{\psi}^\alpha\psi^\beta\overline{\psi}^\gamma\psi^\delta.\end{aligned} \tag{3.160}$$

Then for the last two terms, we can swap the order of the ψ^K to get

$$\begin{aligned}2L =& g_{IJ}\dot{\phi}^I\dot{\phi}^J + i\left(g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J\right) \\ &+ \frac{1}{2}\left(\partial_\gamma\partial_\delta g_{IJ} + \partial_I\partial_J g_{\gamma\delta}\right)\psi^J\overline{\psi}^\delta\psi^\gamma\overline{\psi}^I + g_{IJ}\Gamma^I_{\alpha\beta}\Gamma^J_{\gamma\delta}\psi^\beta\overline{\psi}^\gamma\psi^\delta\overline{\psi}^\alpha.\end{aligned} \tag{3.161}$$

Note that since the metric tensor is symmetric while the $\psi^K$are antisymmetric we can subtract off terms that are equal to zero.

$$\begin{aligned}2L =& g_{IJ}\dot{\phi}^I\dot{\phi}^J + i\left(g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J\right) \\ &+ \underline{\frac{1}{2}\left(\partial_\gamma\partial_\delta g_{IJ} + \partial_I\partial_J g_{\gamma\delta} - \partial_I\partial_\delta g_{\gamma J} - \partial_\gamma\partial_J g_{I\delta}\right)\psi^J\overline{\psi}^\delta\psi^\gamma\overline{\psi}^I} \\ &+ \underline{g_{IJ}\left(\Gamma^I_{\alpha\beta}\Gamma^J_{\gamma\delta} - \Gamma^I_{\alpha\gamma}\Gamma^J_{\beta\delta}\right)\psi^\beta\overline{\psi}^\gamma\psi^\delta\overline{\psi}^\alpha}.\end{aligned} \tag{3.162}$$

Next note that the underline terms are equivalent to the Riemann-Curvature tensor

R_{IJKL} to get

$$2L = g_{IJ}\dot{\phi}^I\dot{\phi}^J + i\left(g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J\right) + R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L. \tag{3.163}$$

Therefore the Lagrangian is

$$L = \frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J + \frac{i}{2}\left(g_{IJ}D_t\overline{\psi}^I\psi^J - g_{IJ}\overline{\psi}^I D_t\psi^J\right) + \frac{1}{2}R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L, \tag{3.164}$$

which is the same as before.

Also the SUSY relationships can be easily derived from

$$\delta\Phi^I = \bar{\epsilon}\mathcal{Q}\Phi^I + \epsilon\overline{\mathcal{Q}}\Phi^I \tag{3.165}$$

where

$$\mathcal{Q} = \partial_\theta + i\overline{\theta}\partial_t \tag{3.166}$$

$$\overline{\mathcal{Q}} = -\partial_{\overline{\theta}} - i\theta\partial_t. \tag{3.167}$$

So having $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ act on the superfield

$$\mathcal{Q}\Phi^I = -\psi^I - i\overline{\theta}F^I + i\overline{\theta}\dot{\phi}^I - i\theta\overline{\theta}\partial_t\psi^I \tag{3.168}$$

$$\overline{\mathcal{Q}}\Phi^I = \overline{\psi}^I - i\theta F^I - i\theta\dot{\phi}^I - i\theta\overline{\theta}\partial_t\overline{\psi}^I \tag{3.169}$$

and taking the variation of the superfield

$$\delta\phi^I + \theta\delta\psi^I + \overline{\theta}\delta\overline{\psi}^I + i\theta\overline{\theta}\delta F^I = \delta\Phi^I, \tag{3.170}$$

we have

$$\begin{aligned}\delta\phi^I + \theta\delta\psi^I + \overline{\theta}\delta\overline{\psi}^I + i\theta\overline{\theta}\delta F^I = \bar{\epsilon}\left(-\psi^I - i\overline{\theta}F^I + i\overline{\theta}\dot{\phi}^I - i\theta\overline{\theta}\partial_t\psi^I\right)\\ +\epsilon\left(\overline{\psi}^I - i\theta F^I - i\theta\dot{\phi}^I - i\theta\overline{\theta}\partial_t\overline{\psi}^I\right).\end{aligned} \tag{3.171}$$

Now equating the coefficients of the fermionic variables θ and $\overline{\theta}$ yields

$$\delta\phi^I = \epsilon\overline{\psi}^I - \overline{\epsilon}\psi^I \tag{3.172}$$

$$\delta\psi^I = \epsilon\left(i\dot{\phi}^I + iF^I\right) \tag{3.173}$$

$$\delta\overline{\psi}^I = \overline{\epsilon}\left(-i\dot{\phi}^J + iF^J\right). \tag{3.174}$$

Now plugging in $F^I = i\Gamma^I_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta$ gives us

$$\delta\phi^I = \epsilon\overline{\psi}^I - \overline{\epsilon}\psi^J \tag{3.175}$$

$$\delta\psi^I = \epsilon\left(i\dot{\phi}^I - \Gamma^I_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\right). \tag{3.176}$$

$$\delta\overline{\psi}^I = \overline{\epsilon}\left(-i\dot{\phi}^J - \Gamma^I_{\alpha\beta}\overline{\psi}^\alpha\psi^\beta\right) \tag{3.177}$$

3.4 Hamiltonian Picture

Now to quantize our system, in general we can find the conjugate momenta as

$$p_I = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^I} = g_{IJ}\dot{\phi}^I \tag{3.178}$$

$$\pi_{\psi^I} = \frac{\partial\mathcal{L}}{\partial\left(\partial_t\psi^I\right)} = ig_{IJ}\overline{\psi}^I \tag{3.179}$$

with the canonical (anti-)commutation relations

$$\left[\phi^I, p_J\right] = i\delta^I_J \tag{3.180}$$

$$\{\psi^I, \overline{\psi}^J\} = g^{IJ}. \tag{3.181}$$

For on S^2, the conjugate momenta are

$$p_1 = r^2\dot{\phi}^1 \tag{3.182}$$

$$p_2 = r^2\sin^2\phi^1\dot{\phi}^2. \tag{3.183}$$

Having the general form of conjugate momenta allows the supercharges to be rewritten in terms on the conjugate momenta p_I given by

$$Q = i\overline{\psi}^I p_I \tag{3.184}$$

$$Q^\dagger = -i\psi^I p_I. \tag{3.185}$$

Doing this for the S^2 supercharges yields the same result.

To find the quantum mechanical version of the Hamiltonian H, we need to consider the ordering of the operators. In order to maintain the desired supersymmetry, we will keep the ordering of the supersymmetric Hamiltonian as

$$\{Q, Q^\dagger\} = 2H. \tag{3.186}$$

Also note that the supercharges Q and $Q^\dagger$ have the opposite F-charge

$$[F, Q] = Q \tag{3.187}$$

$$\left[F, Q^\dagger\right] = -Q^\dagger. \tag{3.188}$$

Consequently, is easy to see that F commutes with the Hamiltonian H

$$[H, F] = 0. \tag{3.189}$$

This means that F is a conserved charge in the quantum theory. Since F generaterms the phase rotation, shown earlier in the Noether charge derivation, we can call this the *femri number operator*.

To finish up the quantization process, we need to specify the representation of the above algebra of observables. Here a natural choice is to use the representation on the space of differential forms,

$$\mathcal{H} = \Omega(M) \otimes \mathbb{C}, \tag{3.190}$$

paired with the Hermitian inner product

$$\langle \omega_1, \omega_2 \rangle = \int_M \overline{\omega}_1 \wedge *\omega_2, \tag{3.191}$$

where $*$ is the Hodge star operator. The observables are represented on this Hilbert space as the operators given by

$$\phi^I = x^I \cdot, \tag{3.192}$$

$$p_I = -i\nabla_I, \tag{3.193}$$

$$\overline{\psi}^I = dx^I \wedge, \tag{3.194}$$

$$\psi^I = g^{IJ} \iota_{\partial/\partial x^J}, \tag{3.195}$$

where the dot represents the act of multiplication, ∇_I is the covariant derivative, dx^I is a differential form, $\wedge$ is an anti-symmetric product, and ι_V is the operation of contraction of the differential form with the vector field V.

To show that these observables can be represented as these operators, all we need to show is that they preserve the (anti-)commutation relationships. For the commutator we have

$$\begin{aligned} \left[\phi^I, p_J\right] &= \left[x^I, -i\nabla_J\right] \\ &= -i\left[x^I, \nabla_J\right] \\ &= -i\left(x^I\nabla_J - \nabla_J x^I\right) \\ &= -i\left(x^I\nabla_J - \delta^I_J - x^I\nabla_J\right) \\ &= i\delta^I_j. \end{aligned} \tag{3.196}$$

For the anti-commutator acting on a p-forms, we have to consider two cases: when

$J \in \{i_1, \ldots, i_p\}$ or when $J \notin \{i_1, \ldots, i_p\}$. So when $J \in \{i_1, \ldots, i_p\}$ we have

$$\begin{aligned}\{\psi^I, \overline{\psi}^J\} dx^{i_1} \wedge \ldots \wedge dx^{i_p} &= \{g^{IK} \iota_{\partial/\partial x^K}, dx^J \wedge\} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\ &= \left(g^{IK} \iota_{\partial/\partial x^K} dx^J \wedge + dx^J \wedge g^{IK} \iota_{\partial/\partial x^K}\right) dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\ &= g^{IK} \left(dx^J \wedge\right) \iota_{\partial/\partial x^K} \left(dx^{i_1} \wedge \ldots \wedge dx^{i_p}\right) \\ &= g^{IK} \delta^J_K (-1)^{j-1} dx^J \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{j-1}} \wedge dx^{i_{j+1}} \wedge \ldots \wedge dx^{i_p} \\ &= g^{IJ} dx^{i_1} \wedge \ldots \wedge dx^{i_p}.\end{aligned} \tag{3.197}$$

Now for when $J \notin \{i_1, \ldots, i_p\}$, we have

$$\begin{aligned}\{\psi^I, \overline{\psi}^J\} dx^{i_1} \wedge \ldots \wedge dx^{i_p} &= \{g^{IK} \iota_{\partial/\partial x^K}, dx^J \wedge\} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\ &= \left(g^{IK} \iota_{\partial/\partial x^K} dx^J \wedge + dx^J \wedge g^{IK} \iota_{\partial/\partial x^K}\right) dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\ &= g^{IK} \delta^J_K dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\ &= g^{IJ} dx^{i_1} \wedge \ldots \wedge dx^{i_p}.\end{aligned} \tag{3.198}$$

Here the supercharges are given as

$$Q = i\overline{\psi}^I p_I = dx^I \wedge \nabla_I = dx^I \wedge \partial_I = d \tag{3.199}$$

$$Q^\dagger = -i\psi^I p_I = -g^{IJ} i_{\partial/\partial x^J} \nabla_I = d^\dagger \tag{3.200}$$

where d is the exterior derivative acting on differential forms and $\partial_I = \frac{\partial}{\partial x^I}$. This makes the Hamiltonian H become

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}\left(dd^\dagger + d^\dagger d\right) = \frac{1}{2}\Delta \tag{3.201}$$

where Δ is the Laplace-Beltrami operator. Thus, the supersymmetric ground states,

or the zero energy states, are simply the harmonic forms

$$\mathcal{H}_{(0)} = \mathcal{H}(M,g) = \bigoplus_{p=0}^{n} \mathcal{H}^p(M,g), \tag{3.202}$$

where $\mathcal{H}(M,g)$ is the space of harmonic forms of the Riemannian manifold (M,g), and $\mathcal{H}^p(M,g)$ is the space of harmonic p-forms.

Now recall that the space of supersymmetric ground states can be characterized as the cohomology of the Q-operator. In the present case, since there is a conserved charge F with

$$[F,Q] = Q, \tag{3.203}$$

the Q-complex and the Q-cohomology are graded by the fermion number

$$F|_{\Omega^P(m)} = p\,\mathrm{Id}. \tag{3.204}$$

Since this is the form-degree and Q is identified as the exterior derivative d, the graded Q-cohomology is the de Rham cohomology [9]

$$\mathcal{H}^p(Q) = H^p_{DR}(M). \tag{3.205}$$

From the general structure of supersymmetric quantum mechanics, we have

$$\mathcal{H}_{(0)} = \mathcal{H}(M,g) \cong H^\bullet(Q) = H^\bullet_{DR}(Q). \tag{3.206}$$

With respect to the F-charge, this refines to

$$\mathcal{H}^p(M,g) \cong H^p_{DR}(Q). \tag{3.207}$$

The supersymmetric index is the Euler characteristic of the Q-complex, namely

$$\mathrm{Tr}(-1)^F = \sum_{p=0}^{n}(-1)^p \mathrm{dim} H^p(Q) = \sum_{p=0}^{n}(-1)^p \mathrm{dim} H^p_{DR}(Q) = \chi(M), \tag{3.208}$$

which is the Euler number of the manifold. Here deformation invariance is the familiar statement that the harmonic forms are equal to the de Rham cohomology classes, which are diffeomorophism invariants [15].

3.4.1 Example: S^2

To calculate the Hamiltonians for S^2, we are going use eq.(3.202). Specifically, since S^2 only depends on two variables, the calculations will include 0,1, and 2-forms. So starting with a 0-form f_0, we have

$$H f_0(x^1, x^2) = \frac{1}{2}\left(dd^\dagger + d^\dagger d\right) f_0, \tag{3.209}$$

where x^1 and x^2 are θ, ϕ respectively. Since we have a 0-form f_0, the $d^\dagger$ will annihilate f_0.

$$H f_0 = \frac{1}{2} d^\dagger d\, f_0 \tag{3.210}$$

Using the definition for d and $d^\dagger$, we have

$$H f_0 = -\frac{1}{2} g^{IK} \nabla_I \partial_J f_0 \iota_{\partial/\partial x^K} dx^J \tag{3.211}$$

Now evaluating the contraction we have

$$H f_0 = -\frac{1}{2} g^{IK} \nabla_I \partial_J f_0 \delta^J_K, \tag{3.212}$$

which further simplfies to

$$H f_0 = -\frac{1}{2} g^{IJ} \nabla_I \partial_J f_0. \tag{3.213}$$

Expanding this out, we have

$$Hf_0 = -\frac{1}{2}g^{IJ}\left(\partial_I\partial_J - \Gamma^K_{IJ}\partial_K\right)f_0. \tag{3.214}$$

Now expanding the summations gives us

$$Hf_0 = -\frac{1}{2}\left[g^{11}\partial_1\partial_1 f_0 + g^{22}(\partial_2\partial_2 f_0 - \Gamma^1_{22}\partial_1 f_0)\right]. \tag{3.215}$$

Using the Connection coefficients from eq.(3.55-56) and the metric tensor from eq.(3.54), we have

$$Hf_0 = -\frac{1}{2}\left[\frac{1}{r^2}\partial_1\partial_1 f_0 + \frac{1}{r^2\sin^2\theta}(\partial_2\partial_2 f_0 + \sin\theta\cos\theta\partial_1 f_0)\right]. \tag{3.216}$$

Simplify this gives us

$$Hf_0 = -\frac{1}{2}\left[\frac{1}{r^2}\partial_1\partial_1 f_0 + \frac{1}{r^2\sin^2\theta}\partial_2\partial_2 f_0 + \frac{1}{r^2\sin\theta}\cos\theta\partial_1 f_0\right]. \tag{3.217}$$

Now we can rewrite $\cos\theta = \partial_1\sin\theta$ and $1 = \frac{\sin\theta}{\sin\theta}$

$$Hf_0 = -\frac{1}{2}\left[\frac{1}{r^2\sin\theta}\sin\theta\partial_1\partial_1 f_0 + \frac{1}{r^2\sin^2\theta}\partial_2\partial_2 f_0 + \frac{1}{r^2\sin\theta}\partial_1\sin\theta\partial_1 f_0\right], \tag{3.218}$$

so that we can undo the product rule for the first and third terms to get

$$Hf_0 = -\frac{1}{2}\left[\frac{1}{r^2\sin\theta}\partial_1\left(\sin\theta\partial_1 f_0\right) + \frac{1}{r^2\sin^2\theta}\partial_2\partial_2 f_0\right]. \tag{3.219}$$

This conveniently works out to be the angular momentum operator squared L^2 in spherical coordinates with constant radial component

$$Hf_0 = \frac{L^2}{2r^2}f_0. \tag{3.220}$$

Now for the 1-forms $f_n(x^1, x^2)dx^n$ where $n \in \{1, 2\}$, we have

$$Hf_n(x^1, x^2)dx^n = \frac{1}{2}\left(dd^\dagger + d^\dagger d\right)f_n dx^n. \tag{3.221}$$

Here we will ignore the Einstein sum convention over the index n since we are only considering the cases when $n = 1$ or $n = 2$ and they are independent of each other. Using the definition for d and $d^\dagger$, we have

$$Hf_n dx^n = -\frac{1}{2}\left[dx^I \wedge \partial_I \left(g^{JK}\iota_{\partial/\partial x^K}\nabla_J\right) + g^{IK}\iota_{\partial/\partial x^K}\nabla_I dx^J \wedge \partial_J\right] f_n dx^n. \quad (3.222)$$

Now distributing the 1-form $f_n dx^n$, gives us

$$Hf_n dx^n = -\frac{1}{2}\left[\partial_I \left(g^{JK}\partial_J f_n\right)\delta^n_K dx^I + g^{IK}\nabla_I \partial_J f_n \iota_{\partial/\partial x^K}\left(dx^J \wedge dx^n\right)\right]. \quad (3.223)$$

This allows us to evaluate the summation over K in the first term and the contractions in the second term to get

$$Hf_n dx^n = -\frac{1}{2}\left[\partial_I \left(g^{Jn}\partial_J f_n\right) dx^I + g^{IK}\nabla_I \partial_J f_n \delta^J_K dx^n - g^{IK}\nabla_I \partial_J f_n \delta^n_K dx^J\right]. \quad (3.224)$$

From here we can evaluate the remain sum over K to get

$$Hf_n dx^n = -\frac{1}{2}\left[\partial_I \left(g^{Jn}\partial_J f_n\right) dx^I + g^{IJ}\nabla_I \partial_J f_n dx^n - g^{In}\nabla_I \partial_J f_n dx^J\right]. \quad (3.225)$$

For $n = 1$, we get

$$Hf_1 dx^1 = -\frac{1}{2}\left[\partial_I \left(g^{J1}\partial_J f_1\right) dx^I + g^{IJ}\nabla_I \partial_J f_1 dx^1 - g^{I1}\nabla_I \partial_J f_1 dx^J\right]. \quad (3.226)$$

Since the metric tensor is symmetric, we must have

$$Hf_1 dx^1 = -\frac{1}{2}\left[g^{11}\partial_I \partial_1 f_1 dx^I + g^{IJ}\nabla_I \partial_J f_1 dx^1 - g^{11}\nabla_1 \partial_J f_1 dx^J\right]. \quad (3.227)$$

Expanding out the covariant derivatives in the first and third terms gives us

$$\begin{aligned} Hf_1 dx^1 = -\frac{1}{2}\Big[&\underline{g^{11}\partial_I \partial_1 f_1 dx^I} + g^{IJ}\nabla_I \partial_J f_1 dx^1 \\ &-g^{11}\left(\underline{\partial_1 \partial_J} - \Gamma^K_{1J}\partial_K\right) f_1 dx^J\Big]. \end{aligned} \quad (3.228)$$

Since partial derivatives can commute with each other, the underlined terms cancel each other.

$$H f_1 dx^1 = -\frac{1}{2}\left[g^{IJ}\nabla_I \partial_J f_1 dx^1 + g^{11}\Gamma^K_{1J}\partial_K f_1 dx^J\right]. \tag{3.229}$$

Expanding the remaining summations in the second term gives us

$$H f_1 dx^1 = -\frac{1}{2}\left[g^{IJ}\nabla_I \partial_J f_1 dx^1 + g^{11}\Gamma^2_{12}\partial_2 f_1 dx^2\right]. \tag{3.230}$$

Using the previous results for the 0-form, the connection coefficients, and metric tensor we get

$$H f_1 dx^1 = \frac{1}{2r^2}\left[L^2 f_1 dx^1 - \cot\theta \partial_2 f_1 dx^2\right]. \tag{3.231}$$

Now for $n = 2$, we have

$$H f_2 dx^2 = -\frac{1}{2}\left[\partial_I\left(g^{J2}\partial_J f_2\right) dx^I + g^{IJ}\nabla_I \partial_J f_2 dx^2 - g^{I2}\nabla_I \partial_J f_2 dx^J\right]. \tag{3.232}$$

Using the product rule for the first term and expanding the covariant derivative in the third term gives us

$$\begin{aligned} H f_2 dx^2 = -\frac{1}{2}\Big[&\left(\partial_I g^{J2}\partial_J + \underline{g^{J2}\partial_I\partial_J}\right) f_2 dx^I + g^{IJ}\nabla_I \partial_J f_2 dx^2 \\ &- g^{I2}\left(\underline{\partial_I\partial_J} - \Gamma^K_{IJ}\partial_K\right) f_2 dx^J\Big]. \end{aligned} \tag{3.233}$$

Since derivatives commute with each other, the underlined terms cancel each other yielding,

$$H f_2 dx^2 = -\frac{1}{2}\left[\partial_I g^{J2}\partial_J f_2 dx^I + g^{IJ}\nabla_I \partial_J f_2 dx^2 + g^{I2}\Gamma^K_{IJ}\partial_K f_2 dx^J\right]. \tag{3.234}$$

Since the metric tensor is symmetric, we must have

$$H f_2 dx^2 = -\frac{1}{2}\left[\partial_I g^{22}\partial_2 f_2 dx^I + g^{IJ}\nabla_I \partial_J f_2 dx^2 + g^{22}\Gamma^K_{2J}\partial_K f_2 dx^J\right]. \tag{3.235}$$

Now expanding out the summations on the third term gives us

$$\begin{aligned} Hf_2dx^2 =& -\frac{1}{2}\left[\partial_I g^{22}\partial_2 f_2 dx^I + g^{IJ}\nabla_I\partial_J f_2 dx^2\right. \\ &\left. +g^{22}\Gamma^1_{22}\partial_1 f_2 dx^2 + g^{22}\Gamma^2_{21}\partial_2 f_2 dx^1\right]. \end{aligned} \tag{3.236}$$

Since $g^{22} = \frac{1}{r^2\sin^2 x^1}$, we must have

$$\begin{aligned} Hf_2dx^2 =& -\frac{1}{2}\left[\underline{\partial_1 g^{22}\partial_2 f_2 dx^1} + g^{IJ}\nabla_I\partial_J f_2 dx^2\right. \\ &\left. +g^{22}\Gamma^1_{22}\partial_1 f_2 dx^2 + \underline{g^{22}\Gamma^2_{21}\partial_2 f_2 dx^1}\right]. \end{aligned} \tag{3.237}$$

Due to metric compatibility, $\nabla_I g^{JK} = 0$, and the connections being symmetric in the lower indices, we can rewrite the underlined terms to get

$$Hf_2dx^2 = -\frac{1}{2}\left[-g^{22}\Gamma^2_{21}\partial_2 f_2 dx^1 + g^{IJ}\nabla_I\partial_J f_2 dx^2 + g^{22}\Gamma^1_{22}\partial_1 f_2 dx^2\right]. \tag{3.238}$$

Using the results for the 0-form, the connections, and metric tensor we have

$$Hf_2dx^2 = \frac{1}{2r^2}\left[\frac{\cot\theta}{\sin^2\theta}\partial_2 f_2 dx^1 + \left(L^2 + \cot\theta\partial_1\right) f_2 dx^2\right]. \tag{3.239}$$

Now for the 2-form $f_{12}(x^1, x^2)dx^1 \wedge dx^2$, we have

$$Hf_{12}(x^1, x^2)dx^1 \wedge dx^2 = \frac{1}{2}\left(dd^\dagger + d^\dagger d\right) f_{12}dx^1 \wedge dx^2. \tag{3.240}$$

Since we are dealing with the surface of a sphere, there are no 3-forms. Therefore, the second term is zero.

$$Hf_{12}dx^1 \wedge dx^2 = \frac{1}{2}dd^\dagger f_{12}dx^1 \wedge dx^2. \tag{3.241}$$

Now using the definition of d and $d^\dagger$, we have

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2}dx^I \wedge \partial_I\left(g^{JK}\iota_{\partial/\partial x^K}\nabla_J\right) f_{12}dx^1 \wedge dx^2. \tag{3.242}$$

Now because of metric compatibility, we have

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2}g^{JK}\partial_I\partial_J f_{12}dx^I \wedge \iota_{\partial/\partial x^K}\left(dx^1 \wedge dx^2\right). \tag{3.243}$$

This allows us to evaluate the contractions to get

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2}g^{JK}\partial_I\partial_J f_{12}\left(\delta^1_K dx^I \wedge dx^2 \wedge -\delta^2_K dx^I \wedge dx^1\right). \tag{3.244}$$

Evaluating the summation over K, gives us

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2}\left[g^{J1}\partial_I\partial_J f_{12}dx^I \wedge dx^2 - g^{J2}\partial_I\partial_J f_{12}dx^I \wedge dx^1\right]. \tag{3.245}$$

Since the metric tensor is symmetric, we must have

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2}\left[g^{11}\partial_I\partial_1 f_{12}dx^I \wedge dx^2 - g^{22}\partial_I\partial_2 f_{12}dx^I \wedge dx^1\right]. \tag{3.246}$$

Now since the wedge product is anti-symmetric, we must have

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2}\left[g^{11}\partial_1\partial_1 f_{12}dx^1 \wedge dx^2 - g^{22}\partial_2\partial_2 f_{12}dx^2 \wedge dx^1\right]. \tag{3.247}$$

Now switching the order of the 1-forms in the second term and fully expanding each term yields

$$Hf_{12}dx^1 \wedge dx^2 = -\frac{1}{2r^2}\left[\partial_1\partial_1 + \frac{1}{\sin^2\theta}\partial_2\partial_2\right] f_{12}dx^1 \wedge dx^2. \tag{3.248}$$

3.5 Deformation by Potential Theory

Now consider modifying the Lagrangian by adding a potential term constructed by a function h such that

$$h : M \to \mathbb{R}. \tag{3.249}$$

This modification is given by addition of

$$\Delta L = -\frac{1}{2} g^{IJ} \partial_I h \partial_J h - D_I \partial_J h \overline{\psi}^I \psi^J \tag{3.250}$$

to the Lagrangian where

$$D_I \partial_J h = \partial_I \partial_J h - \Gamma^K_{IJ} \partial_K h. \tag{3.251}$$

The supersymmetry relation are modified as

$$\delta \phi^I = \epsilon \overline{\psi}^I - \bar{\epsilon} \psi^I \tag{3.252}$$

$$\delta \psi^I = \epsilon \left(i \dot{\phi}^I - \Gamma^I_{\alpha\beta} \overline{\psi}^\alpha \psi^\beta + g^{I\alpha} \partial_\alpha h \right) \tag{3.253}$$

$$\delta \overline{\psi}^I = \bar{\epsilon} \left(-i \dot{\phi}^I - \Gamma^I_{\alpha\beta} \overline{\psi}^\alpha \psi^\beta + g^{I\alpha} \partial_\alpha h \right). \tag{3.254}$$

Similarly, the supercharges are modified as

$$Q_h = \overline{\psi}^I \left(i g_{IJ} \dot{\phi}^J + \partial_I h \right) = \overline{\psi}^I \left(i p_I + \partial_I h \right) \tag{3.255}$$

$$Q_h^\dagger = \psi^I \left(-i g_{IJ} \dot{\phi}^J + \partial_I h \right) = \psi^I \left(-i p_I + \partial_I h \right). \tag{3.256}$$

Under these modifications, the fermion rotation symmetry is preserved and the conserved charge is again

$$F = g_{IJ} \overline{\psi}^I \psi^J. \tag{3.257}$$

As usual, the canonical commutation relation and the same representation of the algebra of variables remains unchanged from our original theory. In particular, the Hilbert space of states is the space of differential forms $\Omega^\bullet(M)$. From here, we can

see that the supercharges are represented as

$$Q_h = d + d\phi^I \wedge \partial_I h = d + dh = e^{-h} d e^h =: d_h \tag{3.258}$$

$$Q_h^\dagger = (d + dh)^\dagger = e^h d^\dagger e^{-h} =: d_h^\dagger. \tag{3.259}$$

The Hamiltonian is chosen so that the supersymmetry relation holds, namely

$$H = \frac{1}{2} Q_h, Q_h^\dagger = \frac{1}{2} \left(d_h d_h^\dagger + d_h^\dagger d_h \right) \tag{3.260}$$

The space of supersymmetric ground states is isomorphic to the cohomology Q-group of the Q_h-operator. Since the conserved fermion number F counts the form-degree, and Q_h has charge 1, the Q-complex and cohomology are graded by the form-degree. However, this Q_h and the Q_0 before the deformation are related by the similarity transformation

$$Q_h = e^{-h} Q_0 e^h, \tag{3.261}$$

and the Q-complex is isomorphic to the old one. Therefore,

$$\mathcal{H}^p_{(0)} \cong \mathcal{H}^p(Q) \cong \mathcal{H}^p(Q_0) = \mathcal{H}^p_{DR}(M). \tag{3.262}$$

In particular, *the dimension of the supersymmetric ground states is independent of the choice of the function h.*

Appendix A

$\mathbb{Z}$-GRADING

In mathematics, a graded space is a space that has the extra structure of a grading or a gradation, which is a decomposition of the space into a direct sum of subspaces. This can be denoted as

$$A = \bigoplus_{i \in \mathcal{I}} A_i \tag{A.1}$$

where A is some space, $A_i \subset A$, and $\mathcal{I}$ is some indexing set.

Now first consider the set of integers $\mathbb{Z} = \mathcal{I}$. This makes A a $\mathbb{Z}$-graded space as

$$A = \bigoplus_{n \in \mathbb{Z}} A_n. \tag{A.2}$$

Next consider an operator $F : A \to A$ defined as F acting on $x \in A$ such that

$$F|_{A_n} = n \,\mathrm{Id}|_{A_n}. \tag{A.3}$$

For example, let $\overline{\psi}_{i_1} \ldots \overline{\psi}_{i_k} |0\rangle = |v_k\rangle \in A_k$, then

$$F\overline{\psi}_{i_1} \ldots \overline{\psi}_{i_k} |0\rangle = k\overline{\psi}_{i_1} \ldots \overline{\psi}_{i_k} |0\rangle = k \, |v_k\rangle \,, \tag{A.4}$$

where $\overline{\psi}_{i_k}$ are fermionic variables. From here we can see that F counts the number of fermionic variables. Now using the fermi number operator F, we construct the a grading such that

$$(-1)^F = \begin{cases} 0, |v_k\rangle \in A_{2n} \\ 1, |v_k\rangle \in A_{2n+1} \end{cases} \tag{A.5}$$

Physically, the even space is associated with the bosons while the odd space is associated with the fermions. This can be denoted as

$$A = A^B \oplus A^F, \tag{A.6}$$

where

$$A^B = \bigoplus_{n\in\mathbb{Z}} A_{2n} \tag{A.7}$$

$$A^F = \bigoplus_{n\in\mathbb{Z}} A_{2n+1}. \tag{A.8}$$

We can see that this is the same as considering the set

$$\mathbb{Z}_n = \mathbb{Z} \setminus n\mathbb{Z}. \tag{A.9}$$

for the special case when $n = 2$ as the indexing set. So if this set had been used initially as the indexing set, the information about which particular subspace A_k the ket $|v_k\rangle$ belonged would have been lost.

To show that the $\mathbb{Z}_2$-graded complex shown in eq.(2.20) splits into a $\mathbb{Z}$-graded complex when

$$[F, Q] = Q, \tag{A.10}$$

first consider when $|v\rangle \in \mathcal{H}^B \oplus \mathcal{H}^F$. By letting the commutator act on $|v\rangle$, we have

$$(FQ - QF)\,|v\rangle = Q\,|v\rangle\,. \tag{A.11}$$

Then decomposing the vector into bosonic and fermionic parts

$$(FQ - QF)\,|v_B\rangle + (FQ - QF)\,|v_F\rangle = Q\,|v_B\rangle + Q\,|v_F\rangle\,. \tag{A.12}$$

Since

$$F|\mathcal{H}^B = 0 \tag{A.13}$$

$$F|\mathcal{H}^F = 1, \tag{A.14}$$

we have

$$FQ\,|v_B\rangle + (FQ - Q)\,|v_F\rangle = Q\,|v_B\rangle + Q\,|v_F\rangle\,. \tag{A.15}$$

Here we can equate coefficient to get

$$FQ\,|v_B\rangle = Q\,|v_B\rangle \tag{A.16}$$

$$(FQ - Q)\,|v_F\rangle = Q\,|v_F\rangle\,. \tag{A.17}$$

Now solving for F, we can see that

$$F|_{Q|v_B\rangle} = 1 = F|_{|v_B\rangle} + 1 \tag{A.18}$$

$$F|_{Q|v_F\rangle} = 2 = F|_{|v_F\rangle} + 1. \tag{A.19}$$

So in a $\mathbb{Z}$-graded hilbert space, a state $|v_k\rangle \in \mathcal{H}_{\|}$ acted on by F is

$$F\,|v_k\rangle = k\,|v_k\rangle\,. \tag{A.20}$$

When it is the case in eq.(A.10), Q acts as

$$Q\,|v_k\rangle = (FQ + QF)\,|v_k\rangle\,. \tag{A.21}$$

By eq.(A.20), we have

$$Q\,|v_k\rangle = (FQ + kQ)\,|v_k\rangle\,. \tag{A.22}$$

Now solving for $FQ\,|v_k\rangle$, we have

$$FQ\,|v_k\rangle = (k+1)\,Q\,|v_k\rangle\,. \tag{A.23}$$

From here, we can see that

$$Q : \mathcal{H}_k \longrightarrow \mathcal{H}_{k+1}, \tag{A.24}$$

giving us the $\mathbb{Z}$-graded complex in eq.(2.26).

Appendix B

WITTEN INDEX

If $\beta \in \mathbb{C}$ such that β scales the Hamiltonian H and maintains being Hermitian, β deforms that theory leading to the invariant, as shown in the calculation below, called the Witten index. To show that the Witten index is an invariant defined in eq.(2.19), first assume the Hamiltonian H is diagonalizable and $\dim\mathcal{H}_{(n)} < \infty$. Since the Hamiltonian H has an eigenvalue of E,

$$(-1)^F e^{-\beta H} |v\rangle = (-1)^F e^{-\beta E} |v\rangle . \tag{B.1}$$

Here β makes the exponent unitless and can vary in meaning depending on the context of the physics the Hamiltonian models. For our purposes, consider β as Wick time on S^1_β. Since the state $|v\rangle$ can be decomposed into a bosonic and a fermionic part, the $\mathbb{Z}_2$-grading assigns a negative sign to the $\mathcal{H}^F$ state.

$$(-1)^F e^{-\beta H} |v\rangle = e^{-\beta E} (|v_B\rangle - |v_F\rangle) . \tag{B.2}$$

Now looking at the trace explicitly, it can be rewritten as a sum of traces restricted to the individual spaces $\mathcal{H}_{(n)}$

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{n=0}^{\infty} \mathrm{Tr}(-1)^F e^{-\beta H}|_{\mathcal{H}_{(n)}}. \tag{B.3}$$

Since the $\mathbb{Z}_2$-grading $(-1)^F$ commutes with the Hamiltonian H, $(-1)^F$ preserves $\mathcal{H}_{(n)}$ exactly as in eq.(2.12). This allows us to decompose $\mathcal{H}_{(n)}$ into a bosonic part and a

fermionic part as in eq.(2.13). Therefore

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{n=0}^{\infty} \mathrm{Tr}\, e^{-\beta H}|_{\mathcal{H}^B_{(n)}} - \sum_{n=0}^{\infty} \mathrm{Tr}\, e^{-\beta H}|_{\mathcal{H}^F_{(n)}}. \tag{B.4}$$

Since the exponentiated Hamiltonian is acting on its eigenstate, we have

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta E_n} \mathrm{Tr}\, \mathrm{Id}|_{\mathcal{H}^B_{(n)}} - \sum_{n=0}^{\infty} e^{-\beta E_n} \mathrm{Tr}\, \mathrm{Id}|_{\mathcal{H}^F_{(n)}}. \tag{B.5}$$

Then evaluating the trace gives us

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta E_n} (\mathrm{dim}\mathcal{H}^B_{(n)} - \mathrm{dim}\mathcal{H}^F_{(n)}). \tag{B.6}$$

Since the bosonic and the fermionic states are paired at each excited energy level, only the ground state with energy E_0 survives:

$$\mathrm{Tr}(-1)^F e^{-\beta H} = e^{-\beta E_0} (\mathrm{dim}\mathcal{H}^B_{(0)} - \mathrm{dim}\mathcal{H}^F_{(0)}). \tag{B.7}$$

Here the ground state energy is $E_0 = 0$, which yields

$$\mathrm{Tr}(-1)^F e^{-\beta H} = (\mathrm{dim}\mathcal{H}^B_{(0)} - \mathrm{dim}\mathcal{H}^F_{(0)}). \tag{B.8}$$

Appendix C

THE HARMONIC OSCILLATOR

In order to find the energy states of the harmonic oscillator where $\omega \geq 0$

$$H = \frac{1}{2}\left(p^2 + \omega^2 x^2\right), \tag{C.1}$$

the Hamiltonian H can be factored to define creation and annihilation operators as

$$a^\dagger = \frac{1}{\sqrt{2\omega}}\left(-ip + \omega x\right) \tag{C.2}$$

$$a = \frac{1}{\sqrt{2\omega}}\left(ip + \omega x\right). \tag{C.3}$$

Now multiplying the two operators together gets

$$a^\dagger a = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2 + i\omega xp - i\omega px\right) \tag{C.4}$$

$$aa^\dagger = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2 - i\omega xp + i\omega px\right). \tag{C.5}$$

Noting that the non-squared terms is a commutator, we have

$$a^\dagger a = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2 + i\omega\left[x, p\right]\right) \tag{C.6}$$

$$aa^\dagger = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2 - i\omega\left[x, p\right]\right). \tag{C.7}$$

Since the canonical commutation relation $[x, p] = i$, we must have

$$a^\dagger a = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2 - \omega\right) \tag{C.8}$$

$$aa^\dagger = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2 + \omega\right). \tag{C.9}$$

Now subtracting eq.(C.8) from eq.(C.9), we get

$$\left[a, a^\dagger\right] = 1. \tag{C.10}$$

Also both eq.(C.8) and eq.(C.9) can be rewritten as

$$\omega\left(a^\dagger a + \frac{1}{2}\right) = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) \tag{C.11}$$

$$\omega\left(aa^\dagger - \frac{1}{2}\right) = \frac{1}{2}\left(p^2 + \omega^2 x^2\right). \tag{C.12}$$

Since the Harmonic oscillator will have a lowest energy state $|E_0\rangle$, this states must be annihilated when acted on by the Hamiltonian. Therefore using eq.(C.1), the Hamiltonian can be expressed in terms of the creation and annihilation operators $a^\dagger, a$ respectively as

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right). \tag{C.13}$$

From here, the Hamiltonian can act on the lowest eigenstate $|E_0\rangle$

$$H\,|E_0\rangle = \omega\left(a^\dagger a + \frac{1}{2}\right)|E_0\rangle. \tag{C.14}$$

Since the system is in its lowest energy state, the state gets annihilated by the operator a. This leaves

$$H\,|E_0\rangle = \frac{1}{2}\omega\,|E_0\rangle. \tag{C.15}$$

Now that the lowest energy value has been found, we can examine how operators $a^\dagger$ and a with the Hamiltonian H. So looking at the commutator for $a^\dagger$ and H

$$\left[H, a^\dagger\right] = Ha^\dagger - a^\dagger H, \tag{C.16}$$

we can use eq.(C.13) to get

$$\left[H, a^\dagger\right] = \omega \left(a^\dagger a + \frac{1}{2}\right) a^\dagger - a^\dagger \omega \left(a^\dagger a + \frac{1}{2}\right), \tag{C.17}$$

which reduces to

$$\left[H, a^\dagger\right] = \omega \left(a^\dagger a a^\dagger - a^\dagger a^\dagger a\right). \tag{C.18}$$

Now using eq.(C.10) in the second term,

$$\left[H, a^\dagger\right] = \omega \left(a^\dagger a a^\dagger - a^\dagger \left(a a^\dagger - 1\right)\right), \tag{C.19}$$

which reduces to

$$\left[H, a^\dagger\right] = \omega a^\dagger. \tag{C.20}$$

Similarly looking at the commutator for a and H

$$[H, a] = Ha - aH, \tag{C.21}$$

we can use eq.(C.13) to get

$$[H, a] = \omega \left(a^\dagger a + \frac{1}{2}\right) a - a\omega \left(a^\dagger a + \frac{1}{2}\right), \tag{C.22}$$

which reduces to

$$[H, a] = \omega \left(a^\dagger a a - a a^\dagger a\right). \tag{C.23}$$

Now using eq.(C.10) in the second term,

$$[H, a] = \omega \left(a^\dagger a a - a \left(a^\dagger a + 1\right)\right), \tag{C.24}$$

which reduces to

$$[H, a] = -\omega a. \tag{C.25}$$

Now consider the Hamiltonian H acting on the eigenstate $a^\dagger |E\rangle$, if we use the commutation relation from eq.(C.20), the states becomes

$$Ha^\dagger |E\rangle = \left(a^\dagger H + \omega a^\dagger\right) |E\rangle . \tag{C.26}$$

Since $|E\rangle$ is an eigenstate of H,

$$Ha^\dagger |E\rangle = \left(Ea^\dagger + \omega a^\dagger\right) |E\rangle . \tag{C.27}$$

Now factoring out the $a^\dagger$

$$Ha^\dagger |E\rangle = (E + \omega) a^\dagger |E\rangle , \tag{C.28}$$

it is the case that H has an eigenvalue for the raise energy state $a^\dagger |E\rangle$. So starting from the lowest energy state, we can construct the "ladder" of energy states by iteratively repeating this calculation from eq.(C.26) to eq.(C.28). Therefore on the n^th time this calculation is repeated, the spectrum of the Hamiltonian can be determined as

$$H |E_n\rangle = \omega \left(n + \frac{1}{2}\right) |E_n\rangle . \tag{C.29}$$

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